1985

# Green's Functions Involving Gauge Transformed Field Variables

Stephen Blair Phillips

GREEN'S FUNCTIONS INVOLVING

GAUGE TRANSFORMED FIELD VARIABLES

by

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Submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy

Faculty of Graduate Studies
The University of Western Ontario
London, Ontario
April 1985

ISBN 0-315-29515-5

# ABSTRACT

The technique of path integration is remarkably well suited for the calculation of scattering amplitudes in relativistic quantum field theories. However, for the case of a field theory which is defined by a Lagrangian density that is invariant under a set of local gauge transformations, the standard Feynman Path Integral is not adequate for the purpose of performing calculations. This inadequacy arises from a problem of multiple counting since equivalent field configurations, which can be connected by a gauge transformation, are considered to be unique and different when the functional integration is performed. The solution to this problem, which maintains the unitarity of the S matrix, involves restricting the functional integration so that only the physical degrees of freedom can contribute to the path integral at each spacetime point.

The major content of this thesis involves a study, in quantum electrodynamics, of connected <u>Green's Functions Involving Gauge Transformed Field Variables</u>. This is accomplished by including the contributions of the unphysical degrees of freedom, which are

associated with the gauge transformation function; in the path integral. There are extra diagrams which contribute to this type of connected Green's function, in addition to those which arise when the corresponding connected Green's function involving the untransformed field variables is considered. The occurrence of these extra diagrams indicates that the term in the gauge field momentum space propagator which is proportional to the tensor $(\alpha-1)p_\mu p_\nu$ can be associated with the propagation of the field associated with the gauge transformation function. The propagation of the physical gauge field does not give rise to any unphysical effects. These arise entirely from the propagation of the unphysical gauge transformation field.

In addition to the work discussed in the main part of this thesis, several other research problems are studied. Pre-Regularization For Supersymmetry, a technique which allows the WTST identities to be explicitly upheld regardless of the regulating technique used, is discussed in Appendix 1. The Longitudinal Contributions to the Vacuum Polarization in the 't Hooft-Veltman Gauge are discussed in Appendix 2 and a calculation of The Four Point Function in N = 4 Supersymmetry is presented in Appendix 3.

## ACKNOWLEDGEMENTS

I would like to extend my sincerest appreciation to Dr. Gerry McKeon for his interest, patience and the freedom and sense of direction that he has given me to pursue what have been, at times, individual research interests during the time that I have worked with him both as a graduate and undergraduate student.

I would also like to extend appreciation to the Natural Sciences and Engineering Research Council of Canada for financial support in the form of a Postgraduate Scholarship.

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# CHAPTER 1 - INTRODUCTION AND SUMMARY OF THE CONVENTIONAL TECHNIQUES FOR THE GENERATION OF CONNECTED GREEN'S FUNCTIONS

To begin the exposition of this thesis it is well worth while to present a comprehensive, but hopefully in the light of what is to follow, informative summary of the understanding of, and techniques for obtaining, the connected Green's functions in a quantized field theory with local gauge invariance. In order to render the calculations to be done later on possible, without the aid of a computer, the gauge field theory chosen for study is massive quantum electrodynamics, with Lagrangian field density,

$$L = -(1/4)F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}(\not\partial - ig\not A)\psi - m\bar{\psi}\psi \qquad (1.1a)$$

This Lagrangian density is invariant under the action of the local gauge transformation:

$$\begin{aligned} A_\mu(x) &\to A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x) \\ \psi_i(x) &\to \psi'_i(x) = \exp\{ig\Lambda(x)\}\psi_i(x) \qquad (1.1b) \\ \bar{\psi}_i(x) &\to \bar{\psi}'_i(x) = \bar{\psi}_i(x)\exp\{-ig\Lambda(x)\} \ . \end{aligned}$$

where $\Lambda(x)$ is the local gauge transformation function.

The ideas presented in this text, as applied to massive quantum electrodynamics, should be generalizable to a more complicated gauge field theory such as quantum chromodynamics. As pointed out by Feynman in a series of papers dealing with the time evolution of quantum mechanical wavefunctions ( 1), and later extended to the spacetime evolution of quantum field theoretical wavefunctionals ( 2), the connected Green's functions, or vacuum to vacuum expectation values for the time ordered product of field operators, are obtainable from the generating functional:

$$Z\{J_\mu, K, \overline{K}\} = \int\{dA_\mu\}\{d\overline{\psi}\}\{d\psi\}\exp\{i\int d^4x( L + L_{source} )\} \tag{1.2a}$$

by functional differentiation as suggested by Schwinger ( 3).
Following multiplication of these connected Green's functions with the appropriate source functions one obtains the associated S matrix elements.

The source Lagrangian density in equation (1.2a) has the form:

$$L_{source} = J_\mu A^\mu + \overline{K}\psi + \overline{\psi}K \tag{1.2b}$$

The connected two point gauge field Green's function is defined by:

$$\langle 0|T(A_\mu(x)A_\nu(y))|0\rangle_c = \frac{\{(1/i^2)(\delta/\delta J^\mu(x))(\delta/\delta J^\nu(y))Z\{J_\mu, K, \overline{K}\}\}}{Z\{J_\mu, K, \overline{K}\}} \tag{1.3}$$

Adopting the policy that perturbation theory in powers of the dimensionless coupling constant, g, makes sense *, the Lagrangian density in equation (1.2) can be split into 2 parts:

$$L = L_o + L_g$$

where $L_g$ is comprised of all the terms in L containing the parameter g.

* "Makes sense" refers to the fact that if perturbation theory in terms of the coupling parameter, g, does not make sense, then it is possible to do perturbation theory that does make sense in terms of an infinitely renormalized running coupling parameter, $\tilde{g}$, by simply replacing g everywhere by $\tilde{g}$.

For this theory,

$$L_o = -(1/4)F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi$$

and $L_g = g\bar{\psi}\not{A}\psi$

As a result of the fact that the exponential of an operator is defined by the power series containing that operator it is useful to write the term in the path integral defined in equation (1.2a) that depends explicitly on the parameter g in the following form:

$$\exp\left\{i\int d^4x\{L_g(A_\mu,\bar{\psi}_i,\psi_j)\}\right\}$$

$$= \sum_{n=0}^{\infty} \frac{\left\{i\int d^4x\{L_g(A_\mu,\bar{\psi}_i,\psi_j)\}\right\}^n}{n(n-1)(n-2).......1} \tag{1.4a}$$

Following Schwinger ( 3), the field variables in the expansion given in equation (1.4a) which are inside the functional integral can be generated by functional differentiation with respect to the source functions in $L_{source}$ that is defined in equation (1.2b). The generating functional that was previously given in equation (1.2a) has the form:

$$Z\{J_\mu,K,\bar{K}\}$$

$$= \int\{dA_\mu\}\{d\bar{\psi}\}\{d\psi\} \sum_{n=0}^{\infty} \frac{\left\{i\int d^4x\{L_g(A_\mu,\bar{\psi}_i,\psi_j)\}\right\}^n}{n(n-1)(n-2)........1} \exp\{i\int d^4x(L_o+L_{source})\} \tag{1.4b}$$

$$= \int\{dA_\mu\}\{d\bar{\psi}\}\{d\psi\} \sum_{n=0}^{\infty} \frac{\left\{i\int d^4x\{L_g((\delta/i\delta J^\mu),(-\delta/i\delta K_i),(\delta/i\delta \bar{K}_j))\}\right\}^n}{n(n-1)(n-2).........1}$$

$$\times \exp\{i\int d^4x(L_o+L_{source})\} \tag{1.4c}$$

$$= \int\{dA_\mu\}\{d\bar{\psi}\}\{d\psi\}Z_g\{J_\mu,K,\bar{K}\}\exp\{i\int d^4x(L_o+L_{source})\} \tag{1,4d}$$

where the quantity $Z_g\{J_\mu,K,\bar{K}\}$ is defined in the following manner:

$$Z_g\{J_\mu,K,\bar{K}\} \equiv \sum_{n=0}^{\infty} \frac{\left\{i\int d^4x\{L_g((\delta/i\delta J^\mu),(-\delta/i\delta K_i),(\delta/i\delta \bar{K}_j))\}\right\}^n}{n(n-1)(n-2)........1} \tag{1.4e}$$

$$\hat{Z}_g\{J_\mu,K,\bar{K}\} = \exp\{i\int d^4x\{L_g((\delta/i\delta J^\mu),(-\delta/i\delta K_i),(\delta/i\delta \bar{K}_j))\}\} \tag{1.4f}$$

The factor $Z_g\{J_\mu,K,\bar{K}\}$ in equation (1.4d), which is independent of the field variables, can be taken outside of the functional integral yielding the result:

$$Z\{J_\mu,K,\bar{K}\} = Z_g\{J_\mu,K,\bar{K}\}\int\{dA_\mu\}\{d\bar{\psi}\}\{d\psi\}\exp\{i\int d^4x(L_o+L_{source})\} \tag{1.5}$$

An attempt can now be made to calculate the generating functional, $Z\{J_\mu,K,\bar{K}\}$, defined in equation (1.5), by converting the integrand into Gaussian form.

$$\begin{aligned}
&\int d^4x\{L_o+L_{source}\} \\
&= \int d^4x\{-(1/4)(\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x))(\partial^\mu A^\nu(x)-\partial^\nu A^\mu(x)) + i\bar{\psi}(x)\not{\partial}\psi(x) \\
&\qquad - m\bar{\psi}(x)\psi(x) + J^\mu(x)A_\mu(x) + \bar{K}(x)\psi(x) + \bar{\psi}(x)K(x)\} \\
&= \int d^4x\{-(1/2)(\partial_\mu A_\nu(x))(\partial^\mu A^\nu(x)) + (1/2)(\partial_\mu A_\nu(x))(\partial^\nu A^\mu(x)) \\
&\qquad + J^\mu(x)A_\mu(x)\} + \int d^4x\ L_{\bar{\psi}\psi} \\
&= \int d^4x\{A_\mu(x)\{((1/2)g_{\alpha\beta}\partial_x^\alpha\partial_x^\beta g^{\mu\nu}-(1/2)\partial_x^\mu\partial_x^\nu)A_\nu(x)\} + J^\mu(x)A_\mu(x)\} \\
&\qquad + \int d^4x\ L_{\bar{\psi}\psi} \\
&= \int d^4x\{A_\mu(x)\{(1/2)\Delta_x^{\mu\nu}A_\nu(x)\} + J^\mu(x)A_\mu(x)\} + \int d^4x\ L_{\bar{\psi}\psi}
\end{aligned} \tag{1.6}$$

where the differential operator $\Delta_x^{\mu\nu}$ is defined by the equation:

$$\Delta^{\mu\nu} \equiv g_{\alpha\beta}\partial_x^\alpha\partial_x^\beta g^{\mu\nu} - \partial_x^\mu\partial_x^\nu \tag{1.7a}$$

and

$$L_{\bar{\psi}\psi} \equiv i\bar{\psi}(x)\not{\partial}\psi(x) - m\bar{\psi}(x)\psi(x) + \bar{K}(x)\psi(x) + \bar{\psi}(x)K(x) \tag{1.7b}$$

Completing the square in equation (1.6) gives the result:++

$$\begin{aligned}
&\int d^4x\{L_o+L_{source}\} \\
&= \int d^4x\{(A_\mu(x)+L_\mu(x))\{(1/2)\Delta_x^{\mu\nu}(A_\nu(x)+L_\nu(x))\} \\
&\qquad - L_\mu(x)((1/2)\Delta_x^{\mu\nu}L_\nu(x))\} + \int d^4x\ L_{\bar{\psi}\psi}
\end{aligned} \tag{1.8}$$

++ The quantity $L_\mu(x)$ is added to the variable $A_\mu(x)$ in order that the functional integral can be converted into a Gaussian form.

provided that

$$\int d^4x \{ L_\mu(x)((1/2)\Delta_x^{\mu\nu}A_\nu(x)) + A_\mu(x)((1/2)\Delta_x^{\mu\nu}L_\nu(x)) \}$$
$$= \int d^4x \{ A_\mu(x)J^\mu(x) \}$$

Integrating by parts, this equation becomes:

$$\int d^4x \{ A_\mu(x)(\Delta_x^{\mu\nu}L_\nu(x) - J^\mu(x)) \} = 0$$

which is identically satisfied if

$$\Delta_x^{\mu\nu}L_\nu(x) = J^\mu(x) \tag{1.9}$$

at all spacetime points, x. It is important to note that the operator $\Delta_x^{\mu\nu}$ has no inverse in the sense of a Green's function in the Fourier transform representation.

Taking the divergence of both sides of equation (1.9) gives the fundamental result

$$\partial_\mu J^\mu(x) = 0 \tag{1.10}$$

which imposes the restriction that, when forming the S matrix element which is associated with a particular connected Green's function, only external sources that are transverse, and hence physical, are to be considered. This is to be contrasted with the more standard approach of using sources with nonzero longitudinal components even though these correspond to unphysical degrees of freedom.

Equation (1.9) can be converted to an integral equation by defining the generalized function $D_{\mu\sigma}(x;y)$ such that,

$$L_\mu(x) = \int d^4y\, D_{\mu\sigma}(x,y)J^\sigma(y) \tag{1.11a}$$

Keeping in consideration the Lorentz and translation invariance of this theory, the generalized function defined in equation (1.11a) has the general form:

$$D_{\mu\sigma}(x,y) = g_{\mu\sigma}\Gamma(x,y) + \{(\partial/\partial x^\mu)(\partial/\partial y^\sigma)\Omega(x,y)\} \tag{1.11b}$$

where $\Gamma(x,y) = \Gamma(x-y)$ and $\Omega(x,y) = \Omega(x-y)$. Using this substitution, equation (1.11a) takes the form:

$$L_\mu(x) = \int d^4y\ \Gamma(x,y) J_\mu(y) \qquad (1.12)$$

following integration by parts on the second term and the use of equation (1.10).

Operating on equation (1.12) with $\Delta_x^{\alpha\mu}$ gives the result:

$$\Delta_x^{\alpha\mu} L_\mu(x) = \int d^4y \{ \{ g_{\rho\sigma} \partial_x^\rho \partial_x^\sigma \Gamma(x,y) \} J^\alpha(y) \} \qquad (1.13a)$$

According to equation (1.9),

$$\Delta^{\alpha\mu} L_\mu(x) = J^\alpha(x),$$

a result that is consistent with equation (1.13a), up to a factor containing the divergence of the source function, $J^\mu(x)$, if

$$g_{\alpha\beta} \partial_x^\alpha \partial_x^\beta \Gamma(x,y) = \delta^4(x-y) \qquad (1.13b)$$

Extensive use will be made of the D'Alembert Green's function, $G(x,y)$, which is defined by the equation:

$$g_{\alpha\beta} \partial_x^\alpha \partial_x^\beta G(x,y) \equiv \delta^4(x-y)$$

so that equation (1.12) can be rewritten in terms of this Green's function in the form:

$$L_\mu(x) = \int d^4y\ G(x,y) J_\mu(y) \qquad (1.14a)$$

where

$$g_{\alpha\beta} \partial_x^\alpha \partial_x^\beta G(x,y) = \delta^4(x-y) \qquad (1.14b)$$

It is interesting then, that even though the middle term in equation (1.8) involves the 'apparently' non-invertible differential operator, $\Delta_x^{\mu\nu}$, the important property of the source function, $J^\mu(x)$, defined in equation (1.10), permits the explicit form of this term to be found. A somewhat more detailed treatment of the mathematical problem associated with inverting this operator is presented in a short publication by the author ( 4).

Even though the mathematical operations leading up to the result in equation (1.14a) are completely correct, the connected Green's functions obtained from this generating functional are not the correct Green's functions for a physical theory since no gauge fixing condition was ever imposed on the gauge field, $A_\mu(x)$, at each spacetime point in the Feynman path integral. As pointed out first by Faddeev and Popov ( 5), and expanded upon by Abers and Lee ( 2), failure to 'fix the gauge' at each and every spacetime point results in the counting of identical field configurations in the functional integral, configurations which are completely equivalent to each other, differing only by a local gauge transformation.

Faddeev and Popov's solution was to insert a Dirac delta functional into the generating functional, with the effect that only those field configurations satisfying the gauge fixing condition would contribute in the path integral, thereby eliminating the problem of multiple counting.

Choosing the covariant gauge fixing term given in the following equation,

$$\partial_\mu A^\mu(x) = 0 \tag{1.15}$$

the generating functional in the Faddeev-Popov formalism has the form:

$$Z^{F.P.}\{J_\mu, K, \bar{K}\} = \int\{dA_\mu\}\{d\bar{\psi}\}\{d\psi\}\delta(\partial_\mu A^\mu)\exp\{i\int d^4x \,(\, L + L_{source}\,)\} \tag{1.16}$$

Choosing the specific representation for the Dirac delta functional,

$$\delta(\partial_\mu A^\mu) = \lim_{\alpha\to 0} \kappa(\alpha)\exp\{\int d^4x\{(-i/2\alpha)(\partial_\mu A^\mu(x))^2\}\} \tag{1.17}$$

where the factor of i is included in the exponent since the time component, $x^o$, is assumed to be imaginary, yields the result,

$$Z^{F.P.}\{J_\mu,K,\bar{K}\} = \kappa(\alpha)\int\{dA_\mu\}\{d\bar{\psi}\}\{d\psi\}\exp\{i\int d^4x\{L+L_{gf}+L_{source}\}\} \quad (1.18)$$

where $L_{gf} = -(1/2\alpha)(\partial_\mu A^\mu(x))^2$, $\kappa(\alpha)$ is some divergent constant and the limit as $\alpha$, the gauge parameter, tends to zero is implied, but not necessarily taken. The presence of the delta functional in the integrand of equation (1.16), which destroys the local gauge invariance of the integrand, forces the introduction of a compensating factor into the integrand that is commonly associated with unphysical fields obeying the wrong statistics. For the case under consideration of an Abelian gauge field theory, these unphysical fields do not couple to the physical fields. Extracting the terms in L containing the coupling constant g, the Faddeev-Popov generating functional can be written, as in equation (1.5), in the form,

$$Z^{F.P.}\{J_\mu,K,\bar{K}\} = \kappa(\alpha)Z_g\{J_\mu,K,\bar{K}\}\int\{dA_\mu\}\{d\bar{\psi}\}\{d\psi\}\exp\{i\int d^4x\{L_o+L_{gf} +L_{source}\}\} \quad (1.19)$$

Repeating the steps leading up to equation (1.8) yields the following result:

$$\begin{aligned}\int d^4x\{L_o+L_{gf}+L_{source}\} &= \int d^4x\{(A_\mu(x)+L_\mu(x))\{(1/2)\tilde{\Delta}^{\mu\nu}_x(A_\nu(x)+L_\nu(x))\} \\ &\quad - L_\mu(x)((1/2)\tilde{\Delta}^{\mu\nu}_x L_\nu(x))\} + \int d^4x\, L_{\bar{\psi}\psi}\end{aligned} \quad (1.20)$$

where the differential operator, $\tilde{\Delta}^{\mu\nu}_x$, is defined according to the equation:

$$\tilde{\Delta}^{\mu\nu}_x \equiv g_{\alpha\beta}\partial^\alpha_x\partial^\beta_x g^{\mu\nu} - \{1-(1/\alpha)\}\partial^\mu_x\partial^\nu_x \quad (1.21)$$

For consistency, the following result must hold:

$$\mathcal{L}_{x}^{\mu\nu} L_{\nu}(x) = J^{\mu}(x) \tag{1.22}$$

However, unlike the situation in equation (1.8), since the differential operator $\mathcal{L}_{x}^{\mu\nu}$ has an inverse in the sense of a Green's function, the explicit form of the right hand side of equation (1.20) can be obtained in a straightforward manner.

Returning to equation (1.1a) which, up until now has defined the Lagrangian field density for massive quantum electrodynamics, it is clearly evident that this Lagrangian density would be just that of two separate, non-interacting fields were it not for the presence of the term $g\bar{\psi}\not{A}\psi$ coupling the fields together. All of the information regarding the dynamics of the electromagnetic interaction is contained in this single term which serves as the lowest order contribution to the fermion-fermion-gauge field connected Green's function.

In order for the two point connected Green's functions for the fermion and gauge fields to represent faithfully the propagation of physical, 'non-interacting' particles, separable from all other matter far enough backward and forward in spacetime, the divergences present in the electron and photon self energies must be argued away. The inhomogeneous Maxwell and Dirac field equations which lead to the specific choices for the two non-interacting Lagrangian field densities from which equation (1.1a) is comprised are,

$$g_{\alpha\beta}\partial_{x}^{\alpha}\partial_{x}^{\beta}A^{\mu}(x) - \partial_{x}^{\mu}\partial_{x}^{\nu}A_{\nu}(x) + \underset{\sim}{J}^{\mu}(x) = 0 \tag{1.23}$$

and

$$i\not{\partial}_{x}\psi(x) - m\psi(x) + \underset{\sim}{K}(x) = 0 \tag{1.24a}$$

and its conjugate equation

$$-i\bar{\psi}(x)\overleftarrow{\not{\partial}}_{x} - m\bar{\psi}(x) + \underset{\sim}{\bar{K}}(x) = 0 \tag{1.24b}$$

In the classical field equations, the sources $\underset{\sim}{J}_{\mu}$, $\underset{\sim}{K}$ and $\underset{\sim}{\overline{K}}$ are externally prescribed perturbations, independent of the field variables themselves. In the limit of vanishing perturbation the fields have simple, undistorted plane wave solutions. Alternatively, it is possible to consider the perturbation sources to be functionals of the respective fields, leading to iterative solutions for the fields in terms of some parameter, $\varepsilon$, such that in the limit as $\varepsilon$ tends to zero, which corresponds to asymptotic states far enough removed in time from the interaction region, the simple plane wave solutions of equations (1.23) and (1.24a) and (1.24b) are recovered.

Furthermore, it is important to note, that equation (1.23) and equations (1.24a) and (1.24b) deal separately with gauge and fermion field variables, respectively. Therefore, in the limit of vanishing minimal coupling, the functionals $\underset{\sim}{J}^{\mu}(x)$, $\underset{\sim}{K}(x)$ and $\underset{\sim}{\overline{K}}(x)$ can contain only the type of field variable(s) according to which of the equations (1.23) or (1.24a) and (1.24b) that they correspond.

The Lagrangian field densities which yield the correct[+] form for the Maxwell and Dirac equations as given in equations (1.23) and (1.24a) and (1.24b) are,

$$L_A = -(1/4)F_{\mu\nu}F^{\mu\nu} + \Delta L_A\{A_\lambda\} \tag{1.25}$$

where $\{(\delta/\delta A_{\mu}(x))\Delta L_A'\{A_\lambda\}\} \equiv \underset{\sim}{J}^{\mu}(x)$ (1.26)

and $L_{\overline{\psi}\psi} = i\overline{\psi}\not{\partial}\psi - m\overline{\psi}\psi + \Delta L_{\overline{\psi}\psi}\{\overline{\psi},\psi\}$ (1.27)

where $\{(\delta/\delta\overline{\psi}_i(x))\Delta L_{\overline{\psi}\psi}\{\overline{\psi},\psi\}\} \equiv \underset{\sim}{K}_i(x)$ (1.28a)

and $\{(\delta/\delta\psi_j(x))\Delta L_{\overline{\psi}\psi}\{\overline{\psi},\psi\}\} \equiv -\underset{\sim}{\overline{K}}_j(x)$ (1.28b)

[+] "Correct" implies that only these Lagrangian field densities with divergent counterterms included yield finite, physically sensible scattering amplitudes.

The correct form of the Lagrangian field density for massive quantum electrodynamics, taking into account the perturbative terms in the Lagrangian densities included in equations (1.25) through (1.28), is not that given in equation (1.1a), but rather,

$$L_{QED} = -(1/4)F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\not{\partial}\psi - m\bar{\psi}\psi + \Delta L_A + \Delta L_{\bar{\psi}\psi} + g\bar{\psi}\not{A}\psi \qquad (1.29)$$

Since the strength of the electromagnetic interaction is governed by the parameter g, in the absence of any interaction the free fields have the simple plane wave solutions obtained by setting $\underset{\sim}{J}_\mu$, $\underset{\sim}{K}$ and $\bar{\underset{\sim}{K}}$ equal to zero in equations (1.23) and (1.24a) and (1.24b). The perturbation parameter, $\varepsilon$, can therefore be identified with some function, $\alpha(g)$, of g, provided that as g tends to zero, which corresponds to a free field state in the limit as t, the time, tends to plus or minus infinity, $\varepsilon$ tends to zero. Furthermore, invariance of the Lagrangian field density in equation (1.29) under the local gauge transformation defined in equation (1.1b) is violated only by those terms in $L_{QED}$ representing non-minimal self coupling of the field variables to themselves. To lowest order in g the local gauge invariance is upheld provided that $\varepsilon \sim O(g^2)$ as g tends to zero, otherwise performing gauge transformations on these contributions would introduce extra terms in the Lagrangian that are linear in g. Suitable choices for the coefficients, which may be divergent, that multiply the bilinear product of field variables in $\Delta L_A$ and $\Delta L_{\bar{\psi}\psi}$, in equation (1.29), result in finite two point connected gauge and fermion field Green's functions.

Clearly, the divergent parts of these coefficients in $\Delta L_A$ and $\Delta L_{\bar{\psi}\psi}$ must compensate all of the divergences which occur in the two point Green's functions that are calculated using the Lagrangian density of equation (1.1a). They will, however, also contain finite terms, whose values must be fixed by physical considerations; for example, the finite term in the two point connected Green's function for the fermion field must give the physical value for the electron mass, as obtained from the position of the pole in the Fourier transform of the propagator.

Having considered this series of arguments as to how the two point connected Green's functions, or propagators, for the fermion and gauge fields, in quantum electrodynamics, can be identified with physical quantities, by a reconsideration of the actual Lagrangian density, the next step is to study more complicated Green's functions, in particular, Green's functions with both external gauge and fermion fields, the simplest of which is the three point fermion-fermion-gauge field connected Green's function. In next to lowest order of perturbation theory, using the Lagrangian density of equation (1.1a), there are four contributions to this connected Green's function as shown in Figure 1:

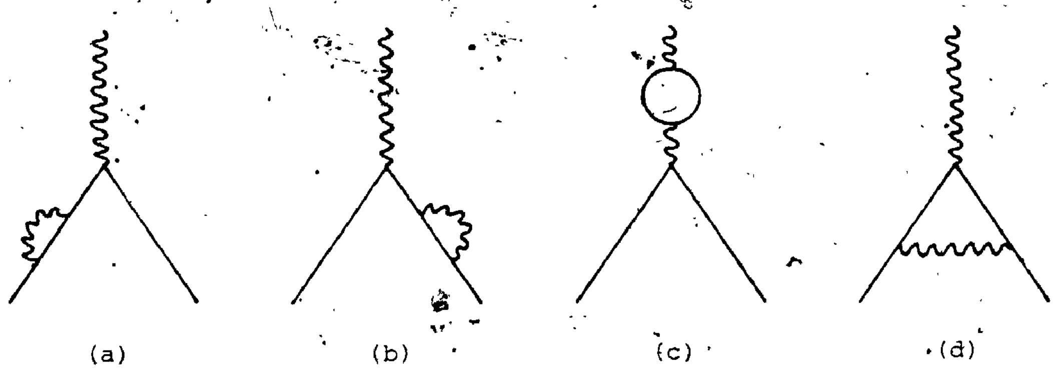


Figure 1. Contributions to the three point connected Green's function in massive quantum electrodynamics using the Lagrangian density of equation (1.1a).

This Green's function is clearly divergent, as all four of the separate contributions are separately divergent, with the divergences failing to cancel amongst themselves. This situation is improved somewhat if the Lagrangian density of equation (1.29) is used to calculate this three point connected Green's function. The seven relevant contributions to this function are shown in Figure 2.

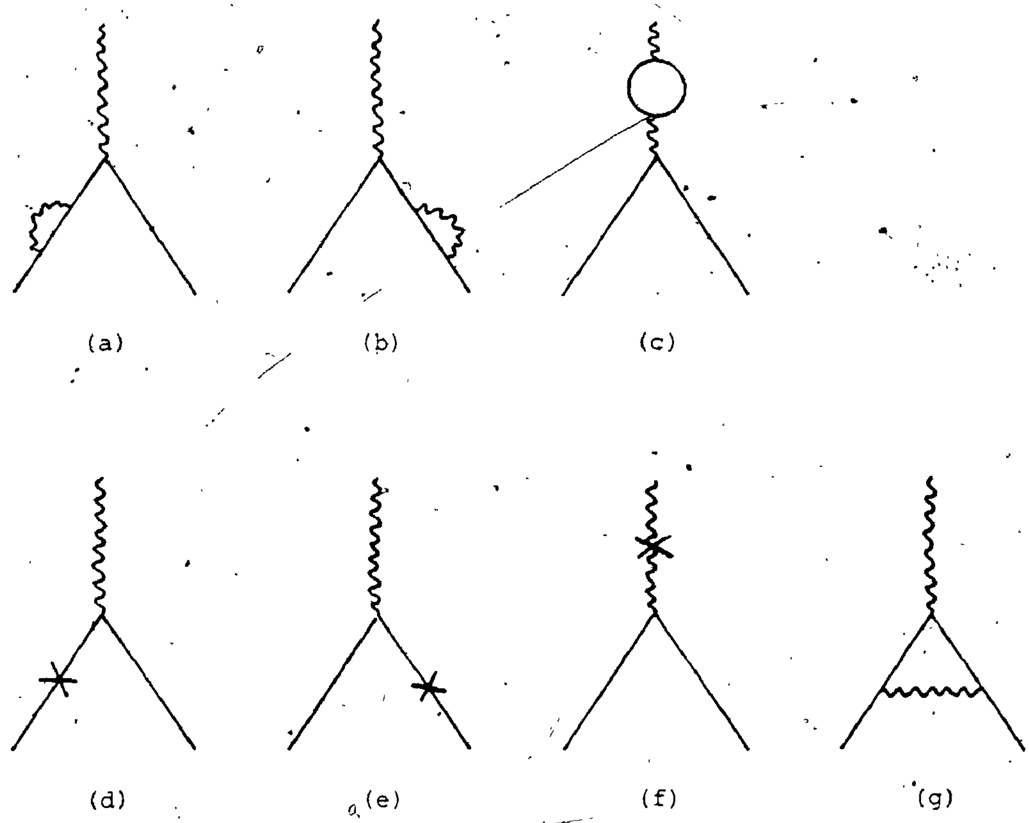


Figure 2. Contributions to the three point connected Green's function in massive quantum electrodynamics using the Lagrangian density of equation (1.29).

By the choice of counterterms made in the study of the two point connected Green's functions, the contribution from Figures 2(a) to 2(f) to the three point Green's function is finite, the only divergent contribution being that contained in Figure 2(g). In light of the discussion regarding the allowable forms of the terms $\Delta L_A$ and $\Delta L_{\bar{\psi}\psi}$ in equation (1.29), in particular, that there are no terms containing both fermion and gauge field variables, there is no mechanism, even using the Lagrangian density of equation (1.29), by which the three point fermion-fermion-gauge field connected Green's function can be rendered finite and thus identifiable with a physical S matrix element, except for the conceptually and operationally complicated techniques of divergent coupling constant renormalization. This renormalization analysis is presented in Bjorken and Drell ( 7).

The main content of this thesis will involve the exposition of an alternative approach to the conventional path integral quantization technique for obtaining connected Green's functions which takes into account the problem of multiple counting of equivalent field configurations, and, as shown by explicit calculation to first order, results in a finite and hence physical three point connected Green's function for massive quantum electrodynamics as described by the Lagrangian density of equation (1.29).

The major difference between the calculations done in this thesis and the calculations that would normally be performed in the Faddeev-Popov formalism rests in the fact that, in this formalism, the integration over the gauge transformation function is actually carried out. In the Faddeev-Popov formalism, after performing an inverse gauge transformation on the functional integration variables, this integration just results in a factor proportional to the 'volume of group space'. This correspondence between the two approaches is expanded upon in chapter 4.

# CHAPTER 2 - EXPOSITION OF AN ALTERNATIVE APPROACH TO PATH INTEGRAL QUANTIZATION

As outlined in the Introduction, the generating functional used for the purpose of obtaining connected Green's functions in a field theory such as massive quantum electrodynamics, involves a sum over all field functionals at each and every spacetime point where the integrand is the phase factor $\exp\{iS\}$, S being the action obtained from the Lagrangian density describing the field theory. This approach is a direct extension of the path integral technique for quantum mechanics, where instead of the wavefunction being the independent degree of freedom at each instant in time as suggested by Feynman ( 1), the wavefunctional at each spacetime point is the independent degree of freedom. For massive quantum electrodynamics, with no gauge fixing condition imposed on the functional integration, the Feynman Path Integral (F.P.I.) has the form given in equation (1.2) with the sources set equal to zero:

$$Z\{J_\mu, K, \bar{K}\}\big|_{J_\mu=K=\bar{K}=0} = \int\{dA_\mu\}\{d\bar{\psi}\}\{d\psi\}\exp\{i\int d^4x\; L_{QED}\} \qquad (2.1)$$

Taking into consideration the discussion presented in the Introduction, the Lagrangian density in equation (2.1) is given by equation (1.29).

The vacuum expectation value of a time ordered product of field operators, for example, the two point connected Green's function for the fermion ~~field~~, has the form, as defined in Abers and Lee ( 2):

$$<0|T(\overline{\psi}_i(x)\psi_j(y))|0>_c = \frac{\int\{dA_\mu\}\{d\overline{\psi}\}\{d\psi\}\overline{\psi}_i(x)\psi_j(y)\exp\{i\int d^4x\ L_{QED}\}}{\int\{dA_\mu\}\{d\overline{\psi}\}\{d\psi\}\exp\{i\int d^4x\ L_{QED}\}} \tag{2.2}$$

The Faddeev-Popov prescription amounts to restricting the functional integration in equation (2.1) such that only those field configurations which satisfy the chosen gauge fixing condition contribute to the generating functional. This is accomplished by inserting a Dirac delta functional, valid at all spacetime points into the integrand of equation (2.1). The common choice is the manifestly covariant condition defined in equation (1.15). Inserting this into equation (2.2) gives the result:

$$<0|T(\overline{\psi}_i(x)\psi_j(y))|0>_c = \frac{\int\{dA_\mu\}\{d\overline{\psi}\}\{d\psi\}\delta(\partial_\nu A^\nu)\overline{\psi}_i(x)\psi_j(y)\exp\{i\int d^4x\ L_{QED}\}}{\int\{dA_\mu\}\{d\overline{\psi}\}\{d\psi\}\delta(\partial_\nu A^\nu)\exp\{i\int d^4x\ L_{QED}\}} \tag{2.3}$$

Formally, at least, in the generating functional of equation (2.3), it is possible to integrate out the gauge field variables by the use of the delta functional, leaving only the two 'real' independent degrees of freedom at each and every spacetime point that are associated with the fermion field variables.

The Lagrangian density for massive quantum electrodynamics, as given in equation ~~(1.29)~~, is invariant, to lowest order in the coupling constant g, under the local gauge transformations on the field variables defined by the equations,

$$A_\mu(x) \to A'_\mu(x) = A_\mu(x) + \partial_\mu\Lambda(x) \qquad (2.4a)$$

$$\psi_i(x) \to \psi'_i(x) = \exp\{ig\Lambda(x)\}\psi_i(x) \qquad (2.4b)$$

$$\bar{\psi}_i(x) \to \bar{\psi}'_i(x) = \bar{\psi}_i(x)\exp\{-ig\Lambda(x)\} \qquad (2.4c)$$

The gauge fixing condition, equation (1.15), as applied to the gauge transformed field variable is:

$$\{(\partial/\partial x^\mu)A^\mu(x)\} + g_{\alpha\beta}\partial^\alpha_x\partial^\beta_x\Lambda(x) = 0 \qquad (2.4d)$$

Instead of using the generating functional given in equation (2.3) for obtaining the vacuum to vacuum transition amplitudes in which there are really only two independent fermionic degrees of freedom at each spacetime point, consider a generating functional in which $L_{QED}$ in the exponent is replaced by its gauge transformed partner, and the functional integration at each spacetime point includes a continuous sum over all values of the transformation function, $\Lambda(x)$, with the restriction now being that only those values of $\Lambda(x)$, such that equation (2.4d) is satisfied, contribute to the generating functional. Since the functional integration over $\Lambda(x)$ can be formally carried out, thereby eliminating the delta functional, there really are now truly independent bosonic degrees of freedom at all spacetime points. By virtue of the manner in which the function $\Lambda(x)$ arises in the path integral; that being as a totally arbitrary gauge transformation function, the integration over $\Lambda(x)$ involves a continuous sum over non-physical degrees of freedom only.

The connected Green's functions are now the vacuum expectation value of the corresponding product of time ordered gauge transformed field operators, and hence, for the two point connected gauge field Green's function, the following equation holds:

$$\langle 0|T\{(A_\mu(x)+\partial_\mu\Lambda(x))(A_\nu(y)+\partial_\nu\Lambda(y))\}|0\rangle_c$$

$$= \frac{\begin{array}{l}\int\{dA_\mu\}\{d\bar\psi\}\{d\psi\}\{d\Lambda\}\delta(\partial_\alpha A^\alpha+g_{\alpha\beta}\partial^\alpha\partial^\beta\Lambda)(A_\mu(x)+\partial_\mu\Lambda(x))\\ \quad\times(A_\nu(y)+\partial_\nu\Lambda(y))\exp\{i\int d^4x\, L_{QED}(A_\mu+\partial_\mu\Lambda,e^{ig\Lambda}\psi,\bar\psi e^{-ig\Lambda})\}\end{array}}{\begin{array}{l}\int\{dA_\mu\}\{d\bar\psi\}\{d\psi\}\{d\Lambda\}\delta(\partial_\alpha A^\alpha+g_{\alpha\beta}\partial^\alpha\partial^\beta\Lambda)\\ \quad\times\exp\{i\int d^4x\, L_{QED}(A_\mu+\partial_\mu\Lambda,e^{ig\Lambda}\psi,\bar\psi e^{-ig\Lambda})\}\end{array}} \tag{2.5}$$

It will be shown, by explicit calculation, that providing the counterterms, $\Delta L_A$ and $\Delta L_{\bar\psi\psi}$, are chosen such that the two point connected Green's functions for the fermion and gauge fields represent the propagators for physical fields, that is, they contain no divergences, that the three point connected Green's function corresponding to the following time ordered product of field operators,

$$\langle 0|T\{(\bar\psi_m(x)e^{-ig\Lambda(x)})(e^{ig\Lambda(y)}\psi_n(y))(A_\mu(v)+\partial_\mu\Lambda(v))\}|0\rangle_c$$

contains no divergences. This is not an unexpected result since, as will be shown, when the gauge parameter $\alpha$ is set equal to zero, which corresponds to the unitary gauge, there is no divergent fermion field wavefunction renormalization and hence, no divergent renormalization of the three point connected Green's function just defined, by virtue of the Q.E.D. vertex Ward Identity (16):

$$Z_1 = Z_2$$

where $Z_1$ is the vertex renormalization factor and $Z_2$ is the fermion field wavefunction renormalization factor.

There are, however, considerably more contributions to the connected Green's functions which must be evaluated. In the case of the three point function there are 19 terms, including the counterterms from $\Delta L_A$ and $\Delta L_{\bar{\psi}\psi}$, instead of the 7 terms illustrated in Figure 2 that arise in the conventional formalism. As commented on earlier, a complete calculation, using this formalism, of all the contributions to this Green's function represents a very laborious task, even for as simple a theory as massive quantum electrodynamics. Therefore, the intent of this work will be to extract the divergences, by means of a convenient regulating technique, and show that they cancel amongst themselves. No attempt is made to actually calculate the finite part of the Green's functions that are considered.

In chapters 4 and 5 detailed calculations are performed for,

i) the gauge and fermion field two point connected Green's functions, and

ii) the fermion-fermion-gauge field three point connected Green's function (including, as a special case, the amplitude for the Rutherford scattering of an electron (or positron) off a fixed Coulomb potential)

using the alternative approach to path integral quantization that is discussed in this chapter, as applied to the Lagrangian density defined in equation (1.29).

An extension of this technique to the case of pure Yang-Mills theories is presented in chapter 6. As a result of the fact that in theories such as this the transformation equation for the gauge field variable contains the coupling parameter, to a given order in g, the sum of certain terms, which are well defined, to a total connected Green's function must vanish. This behaviour is studied in the context of the two point gauge field connected Green's function.

A brief set of conclusions is then presented.

# CHAPTER 3 - FORMATION OF THE GENERATING FUNCTIONAL AND FORMAL DEFINITION OF THE N-POINT CONNECTED GREEN'S FUNCTION

The generating functional to be considered has the explicit form:

$$
\begin{aligned}
Z\{J_1^\mu, J_2^\mu, J, K_1, K_2, \overline{K}_1, \overline{K}_2\} \\
= \int \{dA_\mu\}\{d\overline{\psi}\}\{d\psi\}\{d\Lambda\}\delta(\partial_\nu A^\nu + g_{\alpha\beta}\partial^\alpha\partial^\beta\Lambda) \\
\times \exp\{i\int d^4x \, \{L_{QED}(A_\mu + \partial_\mu\Lambda, e^{ig\Lambda}\psi, \overline{\psi}e^{-ig\Lambda}) \\
+ L_{source}\}\}
\end{aligned}
\tag{3.1}
$$

where

$$
\begin{aligned}
L_{QED} = -(1/4)F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}(\not\partial - ig\not A)\psi - m\overline{\psi}\psi \\
+ g^2\{(A_\mu + \partial_\mu\Lambda)\{\Gamma^{\mu\nu}(A_\nu + \partial_\nu\Lambda)\}\} \\
+ g^2\{\overline{\psi}_i e^{-ig\Lambda}\{\Sigma_{ij}(e^{ig\Lambda}\psi_j)\}\}
\end{aligned}
\tag{3.2}
$$

and

$$
\begin{aligned}
L_{source} = J_1^\mu(A_\mu + \partial_\mu\Lambda) + J_2^\mu A_\mu + J\Lambda + \overline{K}_1 e^{ig\Lambda}\psi + \overline{K}_2\psi \\
+ \overline{\psi}e^{-ig\Lambda}K_1 + \overline{\psi}K_2
\end{aligned}
\tag{3.3}
$$

For the purposes of these calculations, which deal solely with perturbative corrections to 'classical' amplitudes, only the parts which are bilinear in the respective field variables are included in the counterterms, $\Delta L_A$ and $\Delta L_{\overline{\psi}\psi}$. Other terms such as those which are trilinear in the field variables will only be needed when considering higher point functions.

Individual source terms are introduced in the quantity $L_{source}$ defined in equation (3.3) for both the gauge transformed, $(e^{ig\Lambda}\psi, \bar{\psi}e^{-ig\Lambda}, A_\mu+\partial_\mu\Lambda)$, and the untransformed, $(\psi,\bar{\psi},A_\mu)$, fermion and gauge field variables and for the gauge transformation function, $\Lambda$. For ease of calculation these source terms are all linear in the source functions. The term $J\Lambda$ could, in principle, be replaced by a term like $J^2\Lambda^2$. However, if this were done then the functional differentiation with respect to the source J would introduce a complicated factor that is both quadratic in $\Lambda$ and linear in J.

Furthermore, since the physical quantities that are eventually obtained involve taking the limit as all of the sources are set equal to zero, if such a source term were used, it would not be possible to identify the vacuum expectation value of a time ordered product of field operators with a particular functional derivative of the associated generating functional. This equivalence is discussed in Abers and Lee ( 2) and Huang (18).

It is only absolutely necessary to introduce the three source functions $J_1^\mu$, $K_1$ and $\bar{K}_1$. The other source functions, J, $J_2^\mu$, $K_2$ and $\bar{K}_2$, are introduced in order to make the calculations more straightforward since they are coupled in the quantity $L_{source}$ to the actual functional integration variables which occur in the generating functional that is defined in equation (3.1). The perturbative terms in $L_{QED}$ and $L_{source}$ which contain the field variables $\Lambda$, $A_\mu$, $\bar{\psi}$ and $\psi$ are also most easily expressed in terms of functional derivatives with respect to these four auxiliary source functions.

The factors, $\Gamma^{\mu\nu}$ and $\Sigma_{ij}$, are differential operators in space-time in order that translation invariance is maintained.

It is convenient, as in the case of the Faddeev-Popov formalism, to express the Dirac delta functional in the following exponential form:

$$\delta(\partial_\mu A^\mu + g_{\alpha\beta}\partial^\alpha\partial^\beta\Lambda) \equiv \lim_{\alpha\to 0} \kappa(\alpha)\exp\{-(i/2\alpha)\int d^4x\ (\partial_x^\mu A_\mu(x) + g_{\alpha\beta}\partial_x^\alpha\partial_x^\beta\Lambda(x))^2\} \tag{3.4}$$

The generating functional then takes the form:

$$Z\{J_1^\mu, J_2^\mu, J, K_1, K_2, \overline{K}_1, \overline{K}_2\}$$

$$= \lim_{\alpha\to 0} \kappa(\alpha)\int\{dA_\mu\}\{d\overline{\psi}\}\{d\psi\}\{d\Lambda\}\exp\{i\int d^4x\ \{L_{g'} + L_g\}\} \tag{3.5}$$

where

$$\begin{aligned} L_{g'} = {} & -(1/4)F_{\mu\nu}F^{\mu\nu} + i\overline{\psi}\not{\partial}\psi - m\overline{\psi}\psi - (1/2\alpha)(\partial_\mu A^\mu + g_{\alpha\beta}\partial^\alpha\partial^\beta\Lambda)^2 \\ & + J_1^\mu(A_\mu + \partial_\mu\Lambda) + J_2^\mu A_\mu + J\Lambda + \overline{K}_1\psi + \overline{K}_2\psi + \overline{\psi}K_1 + \overline{\psi}K_2 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} L_g = {} & g\overline{\psi}\not{A}\psi + \overline{K}_1(e^{ig\Lambda}-1)\psi + \overline{\psi}(e^{-ig\Lambda}-1)K_1 \\ & + g^2\{(A_\mu + \partial_\mu\Lambda)\{\Gamma^{\mu\nu}(A_\nu + \partial_\nu\Lambda)\}\} \\ & + g^2\{\overline{\psi}_i e^{-ig\Lambda}\{\Sigma_{ij}(e^{ig\Lambda}\psi_j)\}\} \end{aligned} \tag{3.7}$$

As g tends to zero, the coefficients of the bilinear counterterms have the following behaviour:

$$\Gamma^{\mu\nu}(\partial/\partial x^\lambda) \sim O(1)$$

and

$$\Sigma_{ij}(\partial/\partial x^\lambda) \sim O(1)$$

By use of the technique of functional differentiation, the factor containing all powers of g in the generating functional, $\exp\{i\int d^4x\ L_g\}$, can be extracted outside of the functional integral. The generating functional then takes on the 'Gaussian' form:

$$Z\{J_1^\mu, J_2^\mu, J, K_1, K_2, \overline{K}_1, \overline{K}_2\}$$

$$= \exp\{i\int d^4x\ L_g\} \lim_{\alpha\to 0} \kappa(\alpha)\int\{dA_\mu\}\{d\overline{\psi}\}\{d\psi\}\{d\Lambda\}\exp\{i\int d^4x\ L_{g'}\} \tag{3.8}$$

where

$$L_g = L_g\{A_\mu(x)=(\delta/i\delta J_2^\mu(x));\Lambda(x)=(\delta/i\delta J(x));\overline{\psi}_i(x)=(-\delta/i\delta K_{2i}(x));$$
$$\psi_j(x)=(\delta/i\delta\overline{K}_{2j}(x))\} \tag{3.9}$$

A straightforward, but lengthy calculation, which involves completing the squares of the quadratic functionals in $L_{g'}$, and which also makes use of the discussion given in the Introduction with regards to 'inverting' the operator $\Delta_x^{\mu\nu}$, leads to the result that:

$$\int\{dA_\mu\}\{d\overline{\psi}\}\{d\psi\}\{d\Lambda\}\exp\{i\int d^4x\ L_{g'}\} = \tilde{\kappa}(\alpha)\exp\{i\tau(\alpha)\} \tag{3.10}$$

where

$$\begin{aligned}\tau(\alpha) = (1/2)\int d^4x d^4y \Big\{ & J_1^\mu(x)\{(\alpha-1)\{(\partial/\partial x^\mu)(\partial/\partial y^\nu)F(x,y)\} \\ & - g_{\mu\nu}G(x,y)\}J_1^\nu(y) \\ & - 2J_1^\mu(x)\{\{(\partial/\partial x^\mu)(\partial/\partial y^\nu)F(x,y)\} + g_{\mu\nu}G(x,y)\}J_2^\nu(y) \\ & + (\alpha+1)J(x)F(x,y)J(y) - J_2^\mu(x)G(x,y)J_{2\mu}(y) \\ & + 2\alpha J_1^\mu(x)\{(\partial/\partial x^\mu)F(x,y)\}J(y) \\ & - 2J_2^\mu(x)\{(\partial/\partial x^\mu)F(x,y)\}J(y)\Big\} \\ -\int d^4x d^4y \Big\{ & \overline{K}_1(x)S(x,y)K_1(y) + \overline{K}_1(x)S(x,y)K_2(y) \\ & + \overline{K}_2(x)S(x,y)K_1(y) + \overline{K}_2(x)S(x,y)K_2(y)\Big\}\end{aligned} \tag{3.11}$$

The Green's functions in the definition of $\tau(\alpha)$ satisfy the partial differential equations:

$$g_{\alpha\beta}\partial_x^\alpha\partial_x^\beta G(x,y) \equiv \delta^4(x-y) \tag{3.12a}$$

$$g_{\alpha\beta}\partial_x^\alpha\partial_x^\beta g_{\rho\sigma}\partial_x^\rho\partial_x^\sigma F(x,y) \equiv \delta^4(x-y) \tag{3.12b}$$

$$(i\partial\!\!\!/_x - m)_{ij}S_{jk}(x,y) \equiv \delta_{ik}\delta^4(x-y). \tag{3.12c}$$

The connected Green's functions in this theory are defined in terms of the generating functional by the equation:

$$<0|T(\bar{\phi}_m(x)\ldots\phi_n(y)\ldots\beta_\mu(v)\ldots)|0>_c$$

$$= \frac{\lim\limits_{\alpha\to 0}\kappa(\alpha)\int\{dA_\mu\}\{d\bar{\psi}\}\{d\psi\}\{d\Lambda\}\left\{\bar{\phi}_m(x)\ldots\phi_n(y)\ldots\beta_\mu(v)\ldots\right\} \times \exp\{i\int d^4x\,(L_{g'} + L_g)\}}{\lim\limits_{\alpha\to 0}\kappa(\alpha)\int\{dA_\mu\}\{d\bar{\psi}\}\{d\psi\}\{d\Lambda\}\exp\{i\int d^4x\,(L_{g'} + L_g)\}} \qquad (3.13)$$

where, for convenience the following definitions are made:

$$\bar{\phi}_m(x) \equiv \bar{\psi}_m(x)e^{-ig\Lambda(x)} \qquad (3.14a)$$

$$\phi_n(y) \equiv e^{ig\Lambda(y)}\psi_n(y) \qquad (3.14b)$$

$$\beta_\mu(v) \equiv A_\mu(v)+\partial_\mu\Lambda(v) \qquad (3.14c)$$

Again, by use of the functional differentiation technique, this time with respect to the sources $J_1^\mu$, $K_1$ and $\bar{K}_1$, it is possible to rewrite equation (3.13) in the form:

$$<0|T(\bar{\phi}_m(x)\ldots\phi_n(y)\ldots\beta_\mu(v)\ldots)|0>_c$$

$$= \frac{\left\{(-\delta/i\delta K_{1m}(x))\ldots(\delta/i\delta\bar{K}_{1n}(y))\ldots(\delta/i\delta J_1^\mu(v))\ldots\right\} \times Z\{J_1^\mu,J_2^\mu,J,K_1,K_2,\bar{K}_1,\bar{K}_2\}}{Z\{J_1^\mu,J_2^\mu,J,K_1,K_2,\bar{K}_1,\bar{K}_2\}} \qquad (3.15)$$

evaluated at $J_1^\mu=J_2^\mu=J=K_1=K_2=\bar{K}_1=\bar{K}_2=0$.

Inserting the explicit form of $Z\{J_1^\mu,J_2^\mu,J,K_1,K_2,\bar{K}_1,\bar{K}_2\}$, as given in equations (3.8), (3.10) and (3.11), into equation (3.15) yields the following result for the connected Green's function under consideration:

$$\langle 0|T(\overline{\phi}_m(x)\ldots\phi_n(y)\ldots\beta_\mu(v)\ldots)|0\rangle_c = \frac{\left\{(-\delta/i\delta K_{1m}(x))\ldots(\delta/i\delta\overline{K}_{1n}(y))\ldots(\delta/i\delta J_1^\mu(v))\ldots\right\} \times \exp\{i\int d^4x\, L_g\} \lim_{\alpha\to 0} \kappa(\alpha)\tilde{\kappa}(\alpha)\exp\{i\tau(\alpha)\}}{\exp\{i\int d^4x\, L_g\} \lim_{\alpha\to 0} \kappa(\alpha)\tilde{\kappa}(\alpha)\exp\{i\tau(\alpha)\}} \tag{3.16}$$

The factor $\kappa(\alpha)\tilde{\kappa}(\alpha)$, even though it becomes singular when the limit as $\alpha$ tends to zero is taken, can be cancelled out between the numerator and denominator since it is independent of the source functions after the functional integration variables have been shifted in order to perform the Gaussian functional integral. The limit in equation (3.16) may now be taken, giving, for the connected Green's function, the final result:

$$\langle 0|T(\overline{\phi}_m(x)\ldots\phi_n(y)\ldots\beta_\mu(v)\ldots)|0\rangle_c = \frac{\left\{(-\delta/i\delta K_{1m}(x))\ldots(\delta/i\delta\overline{K}_{1n}(y))\ldots(\delta/i\delta J_1^\mu(v))\ldots\right\} \times \exp\{i\int d^4x\, L_g\}\exp\{i\tau\}}{\exp\{i\int d^4x\, L_g\}\exp\{i\tau\}} \tag{3.17}$$

where, as before, the right hand side of this equation is evaluated with all sources set equal to zero, and from equation (3.11),

$$\begin{aligned} \tau &\equiv \tau(0) \\ &= (1/2)\int d^4x d^4y \Big\{- J_1^\mu(x) D_{\mu\nu}(x,y) J_1^\nu(y) - 2J_1^\mu(x) D_{\mu\nu}(x,y) J_2^\nu(y) \\ &\qquad + J(x)F(x,y)J(y) - J_2^\mu(x)G(x,y)J_{2\mu}(y) \\ &\qquad - 2J_2^\mu(x)\{(\partial/\partial x^\mu)F(x,y)\}J(y)\Big\} \\ &\quad -\int d^4x d^4y \Big\{\overline{K}_1(x)S(x,y)K_1(y) + \overline{K}_1(x)S(x,y)K_2(y) \\ &\qquad + \overline{K}_2(x)S(x,y)K_1(y) + \overline{K}_2(x)S(x,y)K_2(y)\Big\} \end{aligned} \tag{3.18}$$

The Green's function $D_{\mu\nu}(x,y)$, for the purposes of this work, is defined by the equation:

$$D_{\mu\nu}(x,y) \equiv g_{\mu\nu}G(x,y) + \{(\partial/\partial x^{\mu})(\partial/\partial y^{\nu})F(x,y)\} \qquad (3.19)$$

# CHAPTER 4 – TRUE INFRARED DIVERGENCES AND THE TWO POINT CONNECTED GREEN'S FUNCTIONS FOR THE FERMION AND GAUGE FIELDS

Calculations for the two point gauge and fermion field connected Green's functions will be presented in this chapter. The gauge field two point Green's function involves just the standard vacuum polarization insertion with a virtual electron-positron loop and its associated local counterterm. The one loop contribution to the fermion field Green's function, however, involves a considerably greater number of terms than just the standard 'self energy' contribution and the associated fermion mass counterterm. These are studied in Chapter 8 of Bjorken and Drell ( 7). Green's functions such as this one receive contributions, in this formalism, from diagrams which contain the two point Green's function, $F(x,y)$, for the field $\Lambda(x)$, the Fourier transform of which has the momentum dependence,

$$F(p^2) = \{1/(p^2)^2\} \tag{4.1a}$$

This result is most easily established by inserting the Fourier transform representation:

$$F(x) \equiv \int d^4p \; e^{-ip\cdot x} \; F(p^2) \tag{4.1b}$$

into equation (3.12b), the defining equation for this function.

It is this very high degree of polynomial behaviour in the denominator of $F(p^2)$, which will give rise to true infrared divergences in a four dimensional spacetime when performing closed loop calculations, that effect the cancellation of all but the simple fermion mass divergence in the fermion field two point connected Green's function. The fermion field counterterm in the Lagrangian density is a simple gauge invariant mass counterterm. There is no divergent fermion wave-function renormalization.

As a result of the fact that the functional variable $\Lambda(x)$ only appears in terms containing the external fermion field source functions, $K_1(x)$ and $\bar{K}_1(x)$, in the interaction Lagrangian density, $(L_g)$, it is impossible to construct a physically meaningful diagram with an external $\Lambda$ line. Thus, all diagrams with external $\Lambda$ lines are excluded. There is no problem of having to 'invent' some new particle since it is only the effect of $\Lambda(x)$ on Green's functions with external fermion and gauge field lines that is observed.

For the purposes of these calculations the ultraviolet divergent integrals that occur are cut off towards the upper limit of integration of the radial momentum variable, p, at a value $\Lambda$. In order to ascertain a suitable technique for the regulation of the infrared divergent integrals, the 'Cartesian' variables in these integrals are transformed into four dimensional spherical polar coordinate variables by employing the same transformations that are used for treating ultraviolet divergent integrals. The explicit form of these transformations, which are given in appendix 1 of 't Hooft and Veltman ( 6), are:

$$p_0 = p \sin\theta \sin\phi \sin\psi \tag{4.2a}$$

$$p_1 = p \cos\theta \sin\phi \sin\psi \tag{4.2b}$$

$$p_2 = p \cos\phi \sin\psi \tag{4.2c}$$

$$p_3 = p \cos\psi \tag{4.2d}$$

where

$$-\infty < p_0, p_1, p_2, p_3 < \infty$$

and

$$0 \leq p < \infty \; ; \; 0 \leq \theta \leq 2\pi \; ; \; 0 \leq \phi \leq \pi \; ; \; 0 \leq \psi \leq \pi$$

The measure of integration transforms in the following form:

$$d^4p = p^3 \sin\phi \sin^2\psi \, dp \, d\theta \, d\phi \, d\psi \tag{4.3}$$

Finally, the four dimensional sphere subtends a 'solid angle' with the value,

$$\int_0^{2\pi} d\theta \int_0^{\pi} d\phi \int_0^{\pi} d\psi \, \sin\phi \, \sin^2\psi = 2\pi^2 \tag{4.4}$$

The 'infrared' or low energy divergences result entirely from the integration over the 'radial' momentum variable, p. There are, of course, the standard singularities contained in the angular part of the Jacobian of transformation. However, these have no connection with actual infrared divergences, and will be ignored.

Consider a change of the radial integration variable from p to 1/r. This has the effect of mapping the singularity contained in the integrand at $p = 0$ into a singularity of the new integrand at $r = \infty$. By a simple power counting analysis, an integrand which diverges like $p^{-n}$ as p tends to zero maps into an integrand which diverges like $r^{n-2}$ as r tends to infinity. For the case of $n = 1$, a logarithmic singularity at the origin in the original integral maps into a logarithmic divergence at infinity in the transformed integral.

A linearly divergent integral at the origin, corresponding to $n = 2$, maps into a linearly divergent integral at infinity, and so on for higher values of n. From a calculational standpoint, instead of actually changing the radial variable in infrared divergent integrals from p to $1/r$, thereby mapping divergences at the origin into divergences at infinity, in direct analogy with the cut-off treatment of ultraviolet divergent integrals, the lower limit of integration on the radial variable is replaced by $\mu^2/\Lambda$, where $\mu$ is some mass parameter introduced to keep everything dimensionally correct. It is of course mathematically possible to consider, instead of the change of variable, $p = 1/r$, the general change $p = r^{-n}$ where n is some positive number. The problem with this transformation though, for all values of n not equal to one, is that the type of divergences contained in the original and transformed integrals are dimensionally different. The 'symmetric' choice with $n = 1$ upholds the symmetry in the type of divergence, and since it is the presence of these infrared divergent integrals which help to render the Green's functions 'less divergent', it is important that the type of singularity remain unchanged.

For the purpose of illustrating all of the salient features that such an analysis involves, consider the integral defined by the following equation:

$$I \equiv \int d^4p \, \{1/(p^2)^2\{(p+q)^2-m^2\}\} \tag{4.5}$$

Before converting the integration variables in equation (4.5) into spherical polar coordinate variables, the Wick rotation ( 7) must be performed on the variables $p_\alpha$ ( $\alpha$ = 0,1,2,3 ) since, as defined in equation (4.5), the variable $p_0$ is in fact purely imaginary, being integrated between $-i\infty$ and $i\infty$. Setting,

$$p_\alpha = \sum_\alpha \{1 + (i-1)\delta_{\alpha 0}\} k_\alpha \tag{4.6}$$

and likewise for $q_\alpha$, the integral in equation (4.5) takes the form:

$$I = i\int d^4k \cdot \{1/(-k^2)^2\{-(k+\tilde{q})^2-m^2\}\} \tag{4.7}$$

where all of the $k_\alpha$ are real variables.

Rotating the k coordinate system so that the $k_3$ axis is parallel to the fixed vector $\tilde{q}$, in which case the following result is valid:

$$k\cdot\tilde{q} = k\tilde{q}\kappa(\theta,\phi,\psi) \tag{4.8a}$$

where $\kappa(\theta,\phi,\psi) = \cos\psi$ (4.8b)

equation (4.7) becomes,

$$I = -\{i/(2\pi)^4\}\int_\Omega d\Omega\int_0^\infty \{k^3dk/(k^2)^2\{k^2+2k\tilde{q}\kappa+\tilde{q}^2+m^2\}\} \tag{4.9}$$

where $d\Omega$ contains all the angular dependence in the transformed integration measure. By simple power counting it is evident that equation (4.9) involves only an infrared divergence.

It is useful, then, to split the radial integral in equation (4.9) into two parts at some value, $\mu$. Equation (4.9) takes the form,

$$I = -\{i/(2\pi)^4\}\int_\Omega d\Omega\left\{\int_0^\mu \{dk/k\{k^2+2k\tilde{q}\kappa+\tilde{q}^2+m^2\}\} + \text{finite}\right\} \tag{4.10}$$

In order to extract the singularity present in the first term on the right hand side of equation (4.10), perform a Laurent series expansion on the integrand, valid for values of k near zero, keeping only those terms which contribute divergences when integrated. The integrand in equation (4.10) then takes the form:

$$\{1/k\{k^2+2k\tilde{q}\kappa+\tilde{q}^2+m^2\}\} = \{1/k\{\tilde{q}^2+m^2\}\{1+\{(2k\tilde{q}\kappa+k^2)/(\tilde{q}^2+m^2)\}\}\} \tag{4.11a}$$

$$= \{1/k\{\tilde{q}^2+m^2\}\}\{1+O(k)\} \tag{4.11b}$$

$$= \{1/k\{\tilde{q}^2+m^2\}\} + O(1) \tag{4.11c}$$

Inserting this result into equation (4.10) and replacing the lower limit of integration, as already discussed, by the cut-off value, $\mu^2/\Lambda$, gives:

$$I = -\{i/(2\pi)^4\}\int_{\Omega} d\Omega \int_{\mu^2/\Lambda}^{\mu} \{dk/k\{\tilde{q}^2+m^2\}\} + \text{finite} \tag{4.12a}$$

$$= -\{i/(2\pi)^4\}(2\pi^2)\{1/\{\tilde{q}^2+m^2\}\}\ln(\Lambda/\mu) + \text{finite} \tag{4.12b}$$

$$= -\{i/8\pi^2\}\{1/\{\tilde{q}^2+m^2\}\}\ln(\Lambda/\mu) + \text{finite} \tag{4.12c}$$

or, performing the inverse Wick rotation:

$$I = \{i/8\pi^2\}\{1/\{q^2-m^2\}\}\ln(\Lambda/\mu) + \text{finite} \tag{4.13}$$

All of the integrals that arise in the calculation of the Green's functions under consideration can be reduced to linear combinations of the following nine integrals:

$$\int d^4p\ \{1/(p^2)^2\} = \{i/4\pi^2\}\ln(\Lambda/\mu) \tag{4.14a}$$

$$\int d^4p\ \{1/\{p^2\{(p+q)^2-m^2\}\}\} = \{i/8\pi^2\}\ln(\Lambda/\mu) + \text{finite} \tag{4.14b}$$

$$\int d^4p\ \{1/\{p^2-m^2\}\{(p+q)^2-m^2\}\} = \{i/8\pi^2\}\ln(\Lambda/\mu) + \text{finite} \tag{4.14c}$$

$$\int d^4p\ \{p_\alpha/\{p^2\{(p+q)^2-m^2\}\}\} = -\{i/16\pi^2\}q_\alpha\ln(\Lambda/\mu) + \text{finite} \tag{4.14d}$$

$$\int d^4p\ \{p_\alpha/\{p^2-m^2\}\{(p+q)^2-m^2\}\} = -\{i/16\pi^2\}q_\alpha\ln(\Lambda/\mu) + \text{finite} \tag{4.14e}$$

$$\int d^4p\ \{p_\alpha p_\beta/\{p^2-m^2\}\{(p+q)^2-m^2\}\} = \{i/16\pi^2\}\Big\{g_{\alpha\beta}\{-(\Lambda^2/4) + m^2\ln(\Lambda/\mu) - (q^2/6)\ln(\Lambda/\mu)\} + (2/3)q_\alpha q_\beta\ln(\Lambda/\mu)\Big\} + \text{finite} \tag{4.14f}$$

$$\int d^4p\ \{1/(p^2)^2\{(p+q)^2-m^2\}\} = \{i/8\pi^2\}\{1/(q^2-m^2)\}\ln(\Lambda/\mu) + \text{finite} \tag{4.14g}$$

$$\int d^4p\ \{p_\alpha/(p^2)^2\{(p+q)^2-m^2\}\} = \text{finite}, \qquad (4.14h)$$

$$\int d^4p\ \{p_\alpha p_\beta/(p^2)^2\{(p+q)^2-m^2\}\} = \{i/32\pi^2\}g_{\alpha\beta}\ln(\Lambda/\mu) + \text{finite} \qquad (4.14i)$$

The integral, that is defined in equation (4.14a) and which has a nonzero value when momentum cut-offs are used to regulate the divergences, vanishes when the technique of dimensional regularization is employed as a calculational tool. As shown by Lee and Milgram (11), such vanishing behaviour of integrals whose integrand is just an even inverse power of the 'radial' momentum variable results from the fact that, in this formalism, the ultraviolet and infrared poles that are present in a given integral are arranged to cancel each other. As a result of this 'cancellation' of ultraviolet and infrared poles, when using dimensional regularization to calculate an integral such as the one that is defined in equation (4.14b), which contains only an ultraviolet divergence when regulated by momentum cut-offs, the 'negative' of the infrared pole that really isn't present is added to the ultraviolet pole, that is present, to form the total divergent term in this integral. A divergent term which behaves like $\ln(\Lambda/\mu)$ as $\Lambda$ tends to infinity in the cut-off scheme is mapped into a simple pole at n = 4 spacetime dimensions in dimensional regularization. This equivalence was first shown by 't Hooft and Veltman (6,13). The identical result for the divergent term(s) in a complete connected Green's function, such as the one defined in equation (4.18), is obtained using either formalism. Different results, however, may be obtained for each individual term that contributes to this complete, connected Green's function. For example, the contributions to this connected Green's function defined in equations (4.18b) and (4.18c).

Making use of the results derived in Chapter 3, in particular equation (3.17), a calculation for the gauge field two point connected Green's function yields the one loop result:

$$\langle 0|T(B_\mu(x)B_\nu(y))|0\rangle_c$$

$$= iD_{\nu\mu}(y,x) \tag{4.15a}$$

$$- 2ig^2\int d^4w\,\{D_\nu{}^\alpha(y,w)\{\Gamma_{\alpha\beta}(\partial/\partial w^\lambda)\}D^\beta{}_\mu(w,x)\}\} \tag{4.15b}$$

$$+ g^2\int d^4w d^4z\,\{\gamma^\alpha_{ij}S_{jk}(w,z)\gamma^\beta_{kl}S_{li}(z,w)D_{\nu\alpha}(y,w)D_{\beta\mu}(z,x)\} \tag{4.15c}$$

Diagrammatically, where the cross in Figure 3(b) represents the local counterterm, $\Gamma_{\alpha\beta}$, this one loop Green's function equation has the form shown in Figure 3.

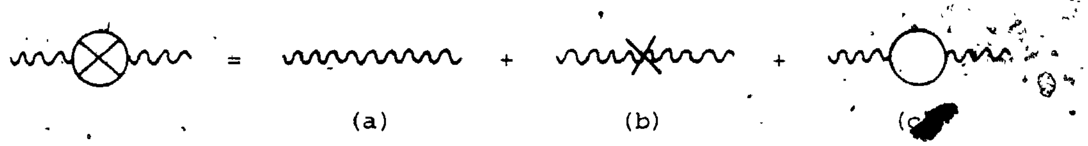

Figure 3. Diagrams contributing to the two point connected gauge field Green's function to one loop order.

Taking for $\Gamma_{\alpha\beta}(\partial/\partial w^\lambda)$ the general form:

$$\Gamma_{\alpha\beta}(\partial/\partial w^\lambda) \equiv g_{\alpha\beta}\Gamma + g_{\alpha\beta}\,\Delta\, g_{\mu\nu}\partial^\mu_w\partial^\nu_w \tag{4.16}$$

where $\Gamma$ and $\Delta$ are divergent constants, the divergent contributions to the two point gauge field Green's function are,

$$(4.15a) = 0$$

$$(4.15b) = g^2\int d^4q\; e^{-iq\cdot(y-x)}D_{\nu\alpha}(q)D_{\beta\mu}(q)\{-2ig^{\alpha\beta}\Gamma+2ig^{\alpha\beta}\Delta q^2\}$$

$$(4.15c) = g^2\int d^4q\; e^{-iq\cdot(y-x)}D_{\nu\alpha}(q)D_{\beta\mu}(q)\Big\{\{ig^{\alpha\beta}\Lambda^2/8\pi^2\} + \{1/6\pi^2\}g^{\alpha\beta}q^2\ln(\Lambda/\mu)\Big\}$$

the sum of which vanishes provided that the constants $\Gamma$ and $\Delta$ satisfy the following equalities:

$$\Gamma = \{\Lambda^2/16\pi^2\} \quad \text{and} \quad \Delta = -\{1/12\pi^2\}\ln(\Lambda/\mu) \tag{4.17}$$

Identical results for these counterterms are given in equations (9.50), (9.54) and (9.64) in Jauch and Rohrlich ( 8). These correspond to mass and wavefunction counterterms, respectively.

A similar calculation for the one loop fermion field two point connected Green's function, the details of which are presented in Appendix 4, yields the result, where the limit as $\alpha$ tends to zero has been taken:

$$\langle 0|T(\overline{\phi}_m(x)\phi_n(y))|0\rangle_c$$

$$= -iS_{nm}(y,x) \tag{4.18a}$$

$$+ \{g^2/2\}F(y,y)S_{nm}(y,x) \tag{4.18b}$$

$$+ \{g^2/2\}F(x,x)S_{nm}(y,x) \tag{4.18c}$$

$$+ ig^2\int d^4w \left\{S_{ni}(y,w)\{\Sigma_{ij}(\partial/\partial w^\lambda)S_{jm}(w,x)\}\right\} \tag{4.18d}$$

$$+ g^2\int d^4w d^4z \; \{G(w,z)S_{nk}(y,z)\gamma_{\mu,kl}S_{li}(z,w)\gamma^\mu_{ij}S_{jm}(w,x)\} \tag{4.18e}$$

$$- ig^2\int d^4w \left\{\{(\partial/\partial w^\mu)F(w,y)\}S_{ni}(y,w)\gamma^\mu_{ij}S_{jm}(w,x)\right\} \tag{4.18f}$$

$$+ ig^2\int d^4w \left\{\{(\partial/\partial w^\mu)F(w,x)\}S_{ni}(y,w)\gamma^\mu_{ij}S_{jm}(w,x)\right\} \tag{4.18g}$$

$$- g^2F(y,x)S_{nm}(y,x) \tag{4.18h}$$

Again, strictly for the purpose of illustration, equation (4.18) has the diagrammatical representation given in Figure 4. The cross in Figure 4(d) represents the local counterterm, $\Sigma_{ij}$, and the double smooth lines represent the 'propagation' of the gauge transformation function, $\Lambda(x)$. In Figures 4(f) and 4(g), and all cases that are similar in the following diagrams, the solid dot on the double smooth line represents the fact that, in configuration space, one end of this line is associated with the untransformed gauge field, $A^\mu(x)$, while the other end is associated with the gauge transformation function, $\Lambda(x)$.

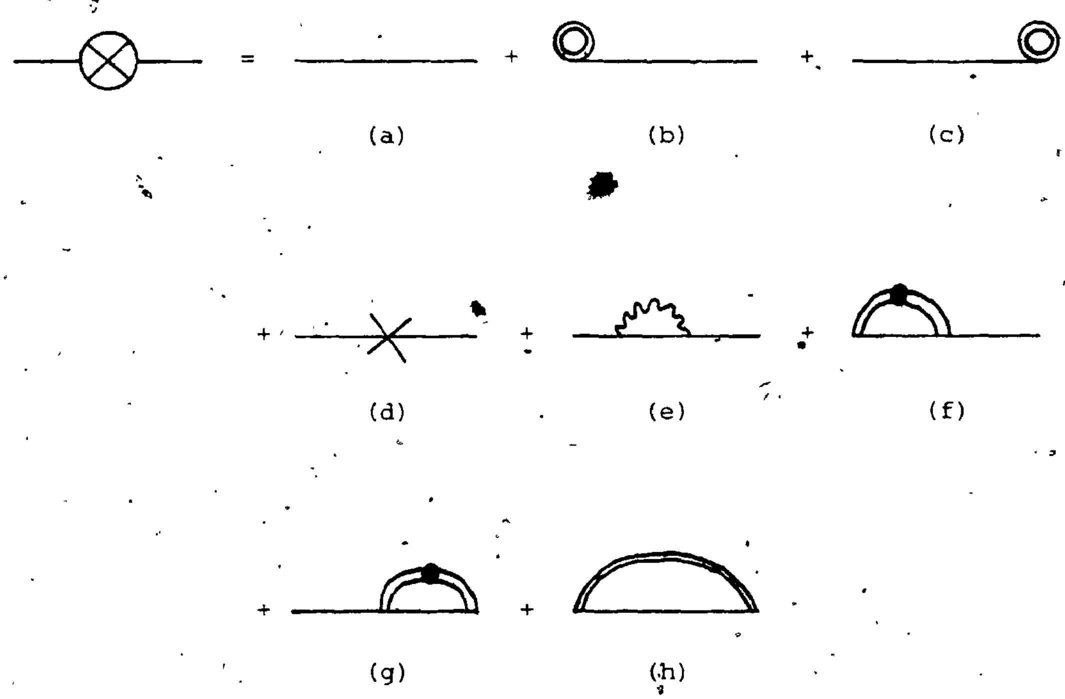


Figure 4. Diagrams contributing to the two point connected fermion field Green's function to one loop order.

As pointed out in the beginning of this chapter, Figures 4(b), 4(c), 4(f), 4(g) and 4(h) actually cancel the divergent term that arises in Figure 4(e) which corresponds to a divergent wavefunction renormalization. In light of this fact then, considering only the divergent terms in the counterterm, $\Sigma_{ij}$ is of the form:

$$\Sigma_{ij}(\partial/\partial w^{\lambda}) \equiv \Sigma\delta_{ij} \qquad (4.19)$$

The divergent contributions to this Green's function are:

(4.18a) = 0

(4.18b) = $I_{nm}(y,x)$

$$(4.18c) = I_{nm}(y,x)$$

$$(4.18d) = g^2\int d^4q\ e^{-iq\cdot(y-x)}\{i\Sigma/(q^2-m^2)^2\}\{(\not q+m)(\not q+m)\}_{nm}$$

$$(4.18e) = g^2\int d^4q\ e^{-iq\cdot(y-x)}\{1/(q^2-m^2)^2\}\{-q^2\not q+2mq^2+7m^2\not q+4m^3\}_{nm} \times\{-i/8\pi^2\}\ln(\Lambda/\mu)$$

$$(4.18f) = -I_{nm}(y,x)$$

$$(4.18g) = -I_{nm}(y,x)$$

$$(4.18h) = -I_{nm}(y,x)$$

where the function $I_{nm}(y,x)$ is defined according to the equation:

$$I_{nm}(y,x) \equiv g^2\int d^4q\ e^{-iq\cdot(y-x)}\{1/(q^2-m^2)\}\{\not q+m\}_{nm}\{i/8\pi^2\}\ln(\Lambda/\mu) \qquad (4.20)$$

The sum of these divergent terms vanishes provided that:

$$\Sigma = \{3m/8\pi^2\}\ln(\Lambda/\mu) \qquad (4.21)$$

This divergent mass counterterm is identical in value to the counterterm given in equation (8.42) of Bjorken and Drell ( 7).

When the contribution to the fermion field two point connected Green's function defined in equation (4.18e) is calculated with the photon two point function, $g^{\mu\nu}G(x,y)$, replaced by the unitary gauge two point function, $D^{\mu\nu}(x,y)$, the resultant self energy contains only a mass divergence. The actual value of the amputated self energy in momentum space, for a general value of the gauge parameter, $\xi$, as calculated using the technique of dimensional regularization, is given in equation (4.22). As pointed out earlier in this chapter, the $1/\bar{\varepsilon}$ poles which occur in dimensional regularization are completely equivalent to the $\ln(\Lambda/\mu)$ singularities which occur when momentum cut-offs are used to regulate the divergences.

$$-i\Sigma(p) = \int d^n k\ (-ig\gamma_\mu)\{i/(\not p-\not k-m)\}(-i/k^2)\{g^{\mu\nu}-\xi k^\mu k^\nu/k^2\}(-ig\gamma_\nu) \tag{4.22a}$$

$$\begin{aligned} = -\{ig^2/(4\pi)^2\}\Big\{&\{-\not p + 4m + \xi(\not p-m)\}(1/\bar\varepsilon) + (-2\not p+6m) \\ &+ a\not p + (\not p-4m)\ln(m^2) + \{\not p(a^2-2a)+4am\}\ln(b) \\ &+\xi\{2(\not p-m) - a\not p + (-\not p+m)\ln(m^2) \\ &- am\ln(b) + (2a-a^2)\not p\ln(b)\}\Big\} \end{aligned} \tag{4.22b}$$

where

$$1/\bar\varepsilon \equiv 1/\{2-(n/2)\} - \gamma_E + \ln(4\pi) \tag{4.22c}$$

$$a \equiv (p^2-m^2)/p^2 \tag{4.22d}$$

and

$$b \equiv (a-1)/a \tag{4.22e}$$

For the unitary gauge, $\xi = 1$, the divergent term proportional to $\not p$, which gives rise to a divergent wavefunction renormalization, vanishes. The equivalence of the result that is obtained by just considering the contribution defined in equation (4.18e), when the unitary gauge is considered, and the result that is obtained from equation (4.18), when all contributions are included, will be established later on in this chapter.

By imposing the restriction of finiteness, and hence identifiability with physical processes, on the two point connected Green's functions that have been calculated in this chapter, the divergent nature of the bilinear counterterms, $\Gamma_{\alpha\beta}(\partial/\partial w^\lambda)$ and $\Sigma_{ij}(\partial/\partial w^\lambda)$ in the Lagrangian field density, has been obtained. These counterterms, however, being of logarithmic behaviour, when momentum cut-offs are used, necessarily require the introduction of a mass scale parameter, $\mu$, the exact value of which, as it arises in the mathematical procedure of calculating the divergent integrals, is completely unconstrained. The only real criterion available for interpreting this mass scale parameter is to consider the physical S matrix elements as being dependent on this unphysical parameter, $\mu$, but subject to a given boundary condition. This is just the scaling hypothesis which forms the basis for the renormalization group equations, as discussed by Wilson and Kogut (14).

The interesting feature of this formalism is that, using these counterterms, a calculation of the three point connected fermion-fermion-gauge field Green's function yields a result that is free of any divergences. The mass scale parameter does occur in the form of an effective coupling constant, $\tilde{g}$, but there is no divergent Lagrangian field density counterterm associated with this vertex.

The unitarity of the S matrix that is associated with the calculation of the two point connected fermion field Green's function just given is most easily studied following a rearrangement of several of the terms which are given in equation (4.18) that contribute to this S matrix, in particular those contributions defined in equations (4.18f), (4.18g) and (4.18h).

Taking as an example the contribution to the connected Green's function given in equation (4.18g), in order that it (and all of the other terms like it) can be interpreted as a true 'propagator' correction, it is necessary to cast it into a form where just a single fermion field propagator leads into the point x. To this end, equation (4.18g) is rewritten in the following form:

$$(4.18g) = ig^2 \int d^4w \left\{ \{ (\partial/\partial w^\mu) F(w,x) \} S_{ni}(y,w) \gamma^\mu_{ij} S_{jm}(w,x) \right\}$$

$$= ig^2\int d^4w d^4z \left\{\{(\partial/\partial w^\mu)F(w,z)\}S_{ni}(y,w)\gamma^\mu_{ij}S_{jl}(w,z)\delta_{lm}\delta^4(z-x)\right\}. \quad (4.23a)$$

Using equation (3.12c), equation (4.23a) can be written in the form:

$$(4.18g) = ig^2\int d^4w d^4z \left\{\{(\partial/\partial w^\mu)F(w,z)\}S_{ni}(y,w)\gamma^\mu_{ij}S_{jl}(w,z) \times \{(i\overrightarrow{\not\partial}_z-m)_{lt}S_{tm}(z,x)\}\right\} \quad (4.23b)$$

and, following one integration by parts with respect to the variable z, in the form:

$$(4.18g) = ig^2\int d^4w d^4z \left\{S_{ni}(y,w)\gamma^\mu_{ij}\left\{\left\{\{(\partial/\partial w^\mu)F(w,z)\}S_{jl}(w,z)\right\} \times \{-i\overleftarrow{\not\partial}_z-m\}_{lt}\right\}S_{tm}(z,x)\right\} \quad (4.23c)$$

$$(4.18g) = ig^2\int d^4w d^4z \left\{S_{ni}(y,w)\gamma^\mu_{ij}\left\{S_{jl}(w,z)\left\{\{(\partial/\partial w^\mu)F(w,z)\}\{-i\overleftarrow{\not\partial}_z\}_{lt}\right\} + \{(\partial/\partial w^\mu)F(w,z)\}\delta_{jt}\delta^4(w-z)\right\}S_{tm}(z,x)\right\} \quad (4.23d)$$

The second term in equation (4.23d) is identically zero as can be seen by representing the function $\{(\partial/\partial w^\mu)F(w,z)\}$ as a Fourier transform, integrating over the variable z thereby setting z equal to w and performing the symmetric integration over the resultant momentum integral, the integrand of which is odd.

The final form for equation (4.18g) is then:

$$(4.18g) = g^2\int d^4w d^4z \left\{S_{ni}(y,w)\gamma^\mu_{ij}S_{jl}(w,z)\{(\partial/\partial w^\mu)(\partial/\partial z^\nu)F(w,z)\} \times \gamma^\nu_{lt}S_{tm}(z,x)\right\} \quad (4.24)$$

Identical analysis leads to the following results for the contributions defined in equations (4.18f) and (4.18h):

$$(4.18f) = g^2\int d^4w d^4z \left\{S_{ni}(y,w)\gamma^\mu_{ij}S_{jl}(w,z)\{(\partial/\partial w^\mu)(\partial/\partial z^\nu)F(w,z)\} \times \gamma^\nu_{lt}S_{tm}(z,x)\right\} \quad (4.25a)$$

$$(4.18h) = - g^2F(w,w)S_{nm}(y,x) - g^2\int d^4w d^4z \left\{S_{ni}(y,w)\gamma^{\mu}_{ij}S_{jl}(w,z)\{(\partial/\partial w^{\mu})(\partial/\partial z^{\nu})F(w,z)\} \times \gamma^{\nu}_{lt}S_{tm}(z,x)\right\} \quad (4.25b)$$

Taking these results into consideration, the two point fermion field connected Green's function, previously defined in equation (4.18), has the rearranged form:

$$<0|T(\bar{\phi}_m(x)\phi_n(y))|0>_c$$

$$= -iS_{nm}(y,x) \quad (4.26a)$$

$$+ ig^2\int d^4w \left\{S_{ni}(y,w)\{\Sigma_{ij}(\partial/\partial w^{\lambda})S_{jm}(w,x)\}\right\} \quad (4.26b)$$

$$+ g^2\int d^4w d^4z \left\{S_{ni}(y,w)\gamma^{\mu}_{ij}S_{jl}(w,z)\gamma^{\nu}_{lt}S_{tm}(z,x)D_{\mu\nu}(w,z)\right\} \quad (4.26c)$$

where the generalized function $D_{\mu\nu}(w,z)$ is defined in equation (3.19).

The two point connected Green's function defined in equation (4.26) is just the one that is obtained using the conventional formalism if the gauge parameter, $\alpha$, that occurs in the gauge fixing term in the Lagrangian field density is set equal to zero, that is, if the singular limit of the unitary gauge is used. The unitarity of this S matrix element is well established, and is shown in many references, for example by 't Hooft and Veltman ( 9) who study the general theory of cutting equations and their connection with the unitarity condition.

In order to tie the results presented herein in with the conventional treatments of the quantization of gauge field theories it is useful to consider the content of Chapter 14 of Abers and Lee ( 2) which deals with an "Intuitive Approach to the Quantization of Gauge Fields".

As first observed by Feynman (10), and later expanded upon by Faddeev and Popov ( 5), in order to perform the quantization of a gauge field theory, it is necessary to restrict the functional integration over the gauge field variable. To accomplish this, a factor such as the one given in equation (14.7) of Abers and Lee ( 2), and reproduced below in equation (4.27), must be inserted into the integrand of the standard functional path integral.

$$1 = \Delta_f\{\vec{A}_\mu\}\int\prod_x dg(x) \prod_{x,a} \delta\{f_a(\vec{A}^g_\mu(x))\} \qquad (4.27)$$

In equation (4.27), which applies to a general non-Abelian gauge field theory, $\vec{A}^g_\mu(x)$ is the gauge transformed gauge field variable and $g(x)$ is the gauge transformation function, previously denoted by $\Lambda(x)$. The point of departure, (or connection, depending on how it is interpreted), between the work presented herein and the conventional analysis as presented by Abers and Lee ( 2) involves the fact that in the conventional treatment, an 'inverse' gauge transformation is performed on the gauge field integration variable, thereby rendering the path integral independent of the continuous summation over the gauge transformation function, $\Lambda(x)$ or $g(x)$, by virtue of the fact that the quantity $\Delta_f\{\vec{A}_\mu\}$, defined in equation (4.27), is independent of the gauge transformation function. This is easiest shown by considering its formal definition, as given in equation (4.27):

$$\Delta_f^{-1}\{\vec{A}_\mu\} \equiv \int\{dg\} \prod_a \delta\{f_a(\vec{A}^g_\mu(x))\} \qquad (4.28a)$$

Performing a local gauge transformation on the gauge field variable, $\vec{A}_\mu(x)$, leads to the following equation:

$$\Delta_f^{-1}\{\vec{A}^{g'}_\mu\} = \int\{dg\} \prod_a \delta\{f_a(\vec{A}^{gg'}_\mu(x))\} \qquad (4.28b)$$

For a specified element, $g'$, of the gauge group, the function $\vec{A}^{gg'}_\mu(x)$ can be considered as having been obtained from the untransformed gauge field variable, $\vec{A}_\mu(x)$, by the action of the single element of the gauge group labelled by $g'' = gg'$, since $g$, $g'$ and $g''$ are all elements of the same gauge group. By a simple 'shift of origin' in the integration over the continuous gauge transformation parameter, $g$, the invariant Hurwitz integration measure in equation (4.28b) can be replaced by the new measure $\{dg''\}$. Equation (4.28b) then takes the form:

$$\Delta_f^{-1}\{\vec{A}_\mu^{g'}\} = \int\{dg''\} \prod_a \delta\{f_a(\vec{A}_\mu^{g''}(x))\} \tag{4.28c}$$

$$= \Delta_f^{-1}\{\vec{A}_\mu\} \tag{4.28d}$$

The work presented in this text involves actually carrying out this integration over the gauge transformation function, and therefore, as is observed, the results should be in complete agreement with those obtained using the conventional methodology. The absence of the factor $\Delta_f\{\vec{A}_\mu\}$ is of no consequence since, for an Abelian gauge field theory, it is independent of the gauge field variable, $A_\mu(x)$.

Referring to equation (3.11), which defines the exponent in the Gaussian generating functional, the interesting content of the work presented herein can be seen. With regard to the calculation of any connected Green's function which involves internal gauge field lines, the entire term in such a propagator which is proportional, in momentum space, to the tensor $p_\mu p_\nu$, has its origin in the vacuum expectation value of one or two of the gauge transformation function field operators. All of the $\alpha$ dependence that is normally present in the term proportional to $p_\mu p_\nu$ in the gauge field two point Green's function is associated with the entirely arbitrary, and hence unphysical, gauge transformation field,

$\Lambda(x)$. Vacuum expectation values of the untransformed gauge field are entirely responsible for the term in an internal gauge field two point Green's function that is proportional to the tensor $g_{\mu\nu}$.

As presented in this work, the association between the vacuum expectation values of the gauge transformation function field operator, which are totally arbitrary, and the terms in an internal two point gauge field Green's function that are proportional to the tensor $p_\mu p_\nu$ helps to explain the fundamental insignificance of such terms when calculating physical amplitudes.

# CHAPTER 5 - THE FERMION-FERMION-GAUGE FIELD THREE POINT CONNECTED GREEN'S FUNCTION AND ITS ASSOCIATED S MATRIX

Making use of the defining equation for the three point connected fermion-fermion-gauge field Green's function, equation (3.17), the following result is obtained:

$$\langle 0|T(\bar{\phi}_m(x)\phi_n(y)B_\mu(v))|0\rangle_c$$

$$= -g\int d^4w\, D_{\alpha\mu}(w,v)S_{ni}(y,w)\gamma^\alpha_{ij}S_{jm}(w,x) \tag{5.1a}$$

$$-\{ig^3/2\}F(y,y)\int d^4w\, D_{\alpha\mu}(w,v)S_{ni}(y,w)\gamma^\alpha_{ij}S_{jm}(w,x) \tag{5.1b}$$

$$-\{ig^3/2\}F(x,x)\int d^4w\, D_{\alpha\mu}(w,v)S_{ni}(y,w)\gamma^\alpha_{ij}S_{jm}(w,x) \tag{5.1c}$$

$$+2g^3\int d^4w d^4z\, S_{ni}(y,w)\gamma^\alpha_{ij}S_{jm}(w,x)D^\beta{}_\mu(z,v)\{\Gamma_{\alpha\beta}(\partial/\partial z^\lambda)G(z,w)\} \tag{5.1d}$$

$$+g^3\int d^4w d^4z\, D_{\alpha\mu}(w,v)S_{nk}(y,z)\{\Gamma_{kl}(\partial/\partial z^\lambda)S_{li}(z,w)\}\gamma^\alpha_{ij}S_{jm}(w,x) \tag{5.1e}$$

$$+g^3\int d^4w d^4z\, D_{\alpha\mu}(w,v)S_{ni}(y,w)\gamma^\alpha_{ij}S_{jk}(w,z)\{\Gamma_{kl}(\partial/\partial z^\lambda)S_{lm}(z,x)\} \tag{5.1f}$$

$$-2ig^3S_{nm}(y,x)\int d^4z\left\{\{\Gamma_{\alpha\beta}(\partial/\partial z^\lambda)D^\beta{}_\mu(z,v)\}\{(\partial/\partial z_\alpha)F(z,y)\}\right\} \tag{5.1g}$$

$$+2ig^3S_{nm}(y,x)\int d^4z\left\{\{\Gamma_{\alpha\beta}(\partial/\partial z^\lambda)D^\beta{}_\mu(z,v)\}\{(\partial/\partial z_\alpha)F(z,x)\}\right\} \tag{5.1h}$$

$$- ig^3\int d^4w d^4z d^4u\ S_{nr}(y,u)\gamma^{\rho}_{rt}S_{tk}(u,z)\gamma_{\alpha,kl}S_{li}(z,w)\gamma^{\alpha}_{ij} \times S_{jm}(w,x)G(w,z)D_{\rho\mu}(u,v) \tag{5.1i}$$

$$- ig^3\int d^4w d^4z d^4u\ S_{nr}(y,u)\gamma_{\beta,rt}S_{tk}(u,z)\gamma^{\beta}_{kl}S_{li}(z,w)\gamma^{\alpha}_{ij} \times S_{jm}(w,x)G(z,u)D_{\alpha\mu}(w,v) \tag{5.1j}$$

$$- ig^3\int d^4w d^4z d^4u\ S_{nr}(y,u)\gamma_{\alpha,rt}S_{tk}(u,z)\gamma^{\beta}_{kl}S_{li}(z,w)\gamma^{\alpha}_{ij} \times S_{jm}(w,x)G(u,w)D_{\beta\mu}(z,v) \tag{5.1k}$$

$$+ ig^3\int d^4w d^4z d^4u\ S_{ni}(y,w)\gamma^{\alpha}_{ij}S_{jm}(w,x)\gamma_{\alpha,kl}S_{lr}(z,u)\gamma^{\rho}_{rt} \times S_{tk}(u,z)G(w,z)D_{\rho\mu}(u,v) \tag{5.1l}$$

$$+ g^3 S_{nm}(y,x)\int d^4w d^4z\ D_{\alpha\mu}(w,v)\{(\partial/\partial z^{\beta})F(z,y)\}\gamma^{\alpha}_{ij} \times S_{jk}(w,z)\gamma^{\beta}_{kl}S_{li}(z,w) \tag{5.1m}$$

$$- g^3\int d^4w d^4z\ D_{\alpha\mu}(w,v)\{(\partial/\partial z^{\beta})F(z,y)\}S_{nk}(y,z)\gamma^{\beta}_{kl} \times S_{li}(z,w)\gamma^{\alpha}_{ij}S_{jm}(w,x) \tag{5.1n}$$

$$- g^3\int d^4w d^4z\ D_{\beta\mu}(z,v)\{(\partial/\partial w^{\alpha})F(w,y)\}S_{nk}(y,z)\gamma^{\beta}_{kl} \times S_{li}(z,w)\gamma^{\alpha}_{ij}S_{jm}(w,x) \tag{5.1o}$$

$$- g^3 S_{nm}(y,x)\int d^4w d^4z\ D_{\alpha\mu}(w,v)\{(\partial/\partial z^{\beta})F(z,x)\}\gamma^{\alpha}_{ij} \times S_{jk}(w,z)\gamma^{\beta}_{kl}S_{li}(z,w) \tag{5.1p}$$

$$+ g^3\int d^4w d^4z\ D_{\alpha\mu}(w,v)\{(\partial/\partial z^{\beta})F(z,x)\}S_{nk}(y,z)\gamma^{\beta}_{kl} \times S_{li}(z,w)\gamma^{\alpha}_{ij}S_{jm}(w,x) \tag{5.1q}$$

$$+ g^3\int d^4w d^4z\ D_{\beta\mu}(z,v)\{(\partial/\partial w^{\alpha})F(w,x)\}S_{nk}(y,z)\gamma^{\beta}_{kl} \times S_{li}(z,w)\gamma^{\alpha}_{ij}S_{jm}(w,x) \tag{5.1r}$$

$$+ ig^3\int d^4w\ F(y,x)D_{\alpha\mu}(w,v)S_{ni}(y,w)\gamma^{\alpha}_{ij}S_{jm}(w,x) \tag{5.1s}$$

In terms of diagrams, equation (5.1) has the form shown in Figure 5.

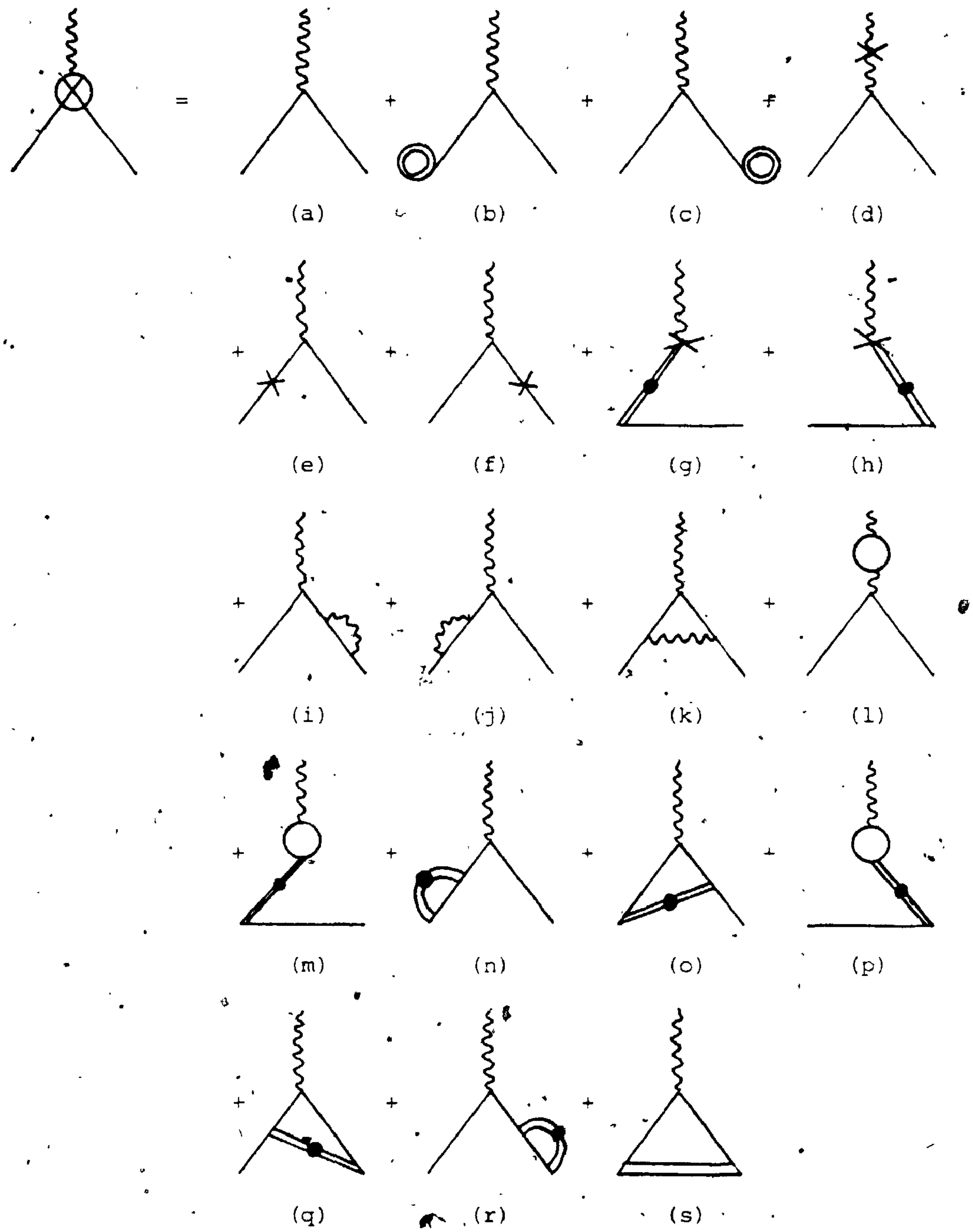


Figure 5. A diagrammatic representation of the fermion-fermion-gauge field three point connected Green's function to one loop order.

Direct use of the integrals given in equation (4.14) yields, for the divergent contributions to the three point Green's function, the following results:

(5.1a) = 0

(5.1b) = $A_{mn\mu}(x,y,v)$

(5.1c) = $A_{mn\mu}(x,y,v)$

(5.1d) + (5.1l) = 0

(5.1e) + (5.1j) = $A_{mn\mu}(x,y,v)$

(5.1f) + (5.1i) = $A_{mn\mu}(x,y,v)$

(5.1g) = 0

(5.1h) = 0

(5.1k) = $-A_{mn\mu}(x,y,v)$

(5.1m) = 0

(5.1n) = $-A_{mn\mu}(x,y,v)$

(5.1o) = 0

(5.1p) = 0

(5.1q) = 0

(5.1r) = $-A_{mn\mu}(x,y,v)$

(5.1s) = $-A_{mn\mu}(x,y,v)$

where the function $A_{mn\mu}(x,y,v)$ is defined by the equation,

$$A_{mn\mu}(x,y,v) = g^3(2\pi)^4\int d^4p\,d^4q\,d^4r\left\{e^{ip\cdot v}\,e^{-iq\cdot y}\,e^{ir\cdot x}\right.$$
$$\times D_{\alpha\mu}(p)S_{ni}(q)\gamma^{\alpha}_{ij}S_{jm}(r)$$
$$\left.\times\,\delta^4(p-q+r)\{1/8\pi^2\}\ln(\Lambda/\mu)\right\} \quad (5.2)$$

The three point fermion-fermion-gauge field connected Green's function contains no divergent terms. Factoring out a propagator on each of the two external fermion field lines and the external gauge field line yields the amputated Green's function which in this case is just the effective coupling strength, $\tilde{g}$. The only difference between this analysis and the conventional treatment is that in the conventional treatment, when the Feynman gauge two point gauge field Green's function is used for photon propagation in internal lines, the amputated three point connected Green's function contains a divergence associated with Figure 5(k), even after the divergent bilinear counterterms are included. It is thus necessary to add to the Lagrangian field density an explicit divergent fermion-fermion-gauge field counterterm. For the calculation presented in this work it is only necessary to add a finite vertex counterterm in order that the effective coupling strength match the observed coupling strength at some value of the scale parameter, $\mu$.

Similar to what occurred in the consideration of the contribution to the fermion field two point connected Green's function defined in equation (4.18e) if, in the contribution to the fermion-fermion-gauge field three point connected Green's function defined in equation (5.1k), the photon two point function for propagation on internal lines, $g^{\mu\nu}G(x,y)$, is replaced by the unitary gauge two point function, $D^{\mu\nu}(x,y)$, then there is no divergent term in this contribution to the vertex function. This ensures that the Q.E.D. vertex Ward Identity, $Z_1 = Z_2$, is satisfied, since, as pointed out in the discussion immediately following equation (4.22), such a replacement also rules out the need for having a divergent fermion field wave-function renormalization factor.

The one loop Green's function that corresponds to Rutherford scattering from a fixed Coulomb potential with no bremsstrahlung emission is obtained from the three point Green's function just calculated by omitting from equation (5.1) all of those terms which contain the effects of a propagating photon, and their respective counterterms. Thus the contributions from equations (5.1d), (5.1g), (5.1h), (5.1l), (5.1m) and (5.1p) are neglected.

In order to obtain explicitly the form of the dependence of the fermion field two point Green's function in the presence of an externally prescribed electromagnetic Coulomb potential, on the Coulomb potential, it is necessary to multiply all of the remaining terms on the right hand side of equation (5.1) by the source function $iJ_1^\mu(v)$ and integrate over all values of the spacetime variable, v. The lowest order term, equation (5.1a), then takes the following form:

$$\Sigma_{nm}(y,x) = -ig\int d^4w\{\int d^4v\; D_{\alpha\mu}(w,v)J_1^\mu(v)\}S_{ni}(y,w)\gamma^\alpha_{ij}S_{jm}(w,x) \tag{5.3}$$

As shown in the Introduction, and considered in the context of forming the S matrix elements for a general connected Green's function later on in this chapter, it is possible to make the following replacement in the integrand of equation (5.3),

$$\tilde{A}_\alpha(w) = - \int d^4v\; \{D_{\alpha\mu}(w,v)J_1^\mu(v)\} \tag{5.4}$$

where

$$\tilde{A}_\alpha(w) = g_{\alpha 0}\{Zg/4\pi|\vec{w}|\} \tag{5.5}$$

for a static point charge with $Zg > 0$, as given in equation (7.4) of Bjorken and Drell (17). This substitution yields the following result for the one loop two point fermion field connected Green's function in the presence of an externally prescribed electromagnetic Coulomb potential, $\tilde{A}_\alpha(w)$:

$$\langle 0|T(\bar{\phi}_m(x)\phi_n(y);\tilde{A}_\alpha(w))|0\rangle_c$$

$$= ig\int d^4w\, \tilde{A}_\alpha(w) S_{ni}(y,w)\gamma^\alpha_{ij} S_{jm}(w,x) \tag{5.6a}$$

$$- \{g^3/2\}F(y,y)\int d^4w\, \tilde{A}_\alpha(w) S_{ni}(y,w)\gamma^\alpha_{ij} S_{jm}(w,x) \tag{5.6b}$$

$$- \{g^3/2\}F(x,x)\int d^4w\, \tilde{A}_\alpha(w) S_{ni}(y,w)\gamma^\alpha_{ij} S_{jm}(w,x) \tag{5.6c}$$

$$- ig^3\int d^4w d^4z\, \tilde{A}_\alpha(w) S_{nk}(y,z)\{\Sigma_{kl}(\partial/\partial z^\lambda) S_{li}(z,w)\}\gamma^\alpha_{ij} S_{jm}(w,x) \tag{5.6d}$$

$$- ig^3\int d^4w d^4z\, \tilde{A}_\alpha(w) S_{ni}(y,w)\gamma^\alpha_{ij} S_{jk}(w,z)\{\Sigma_{kl}(\partial/\partial z^\lambda) S_{lm}(z,x)\} \tag{5.6e}$$

$$- g^3\int d^4w d^4z d^4u\, S_{nr}(y,u)\gamma^\rho_{rt} S_{tk}(u,z)\gamma_{\alpha,kl} S_{li}(z,w)\gamma^\alpha_{ij} \times S_{jm}(w,x) G(w,z)\tilde{A}_\rho(u) \tag{5.6f}$$

$$- g^3\int d^4w d^4z d^4u\, S_{nr}(y,u)\gamma_{\beta,rt} S_{tk}(u,z)\gamma^\beta_{kl} S_{li}(z,w)\gamma^\alpha_{ij} \times S_{jm}(w,x) G(z,u)\tilde{A}_\alpha(w) \tag{5.6g}$$

$$- g^3\int d^4w d^4z d^4u\, S_{nr}(y,u)\gamma_{\alpha,rt} S_{tk}(u,z)\gamma^\beta_{kl} S_{li}(z,w)\gamma^\alpha_{ij} \times S_{jm}(w,x) G(u,w)\tilde{A}_\beta(z) \tag{5.6h}$$

$$+ ig^3\int d^4w d^4z\, \tilde{A}_\alpha(w)\{(\partial/\partial z^\beta)F(z,y)\} S_{nk}(y,z)\gamma^\beta_{kl} S_{li}(z,w) \times \gamma^\alpha_{ij} S_{jm}(w,x) \tag{5.6i}$$

$$+ ig^3\int d^4w d^4z\, \tilde{A}_\beta(z)\{(\partial/\partial w^\alpha)F(w,y)\} S_{nk}(y,z)\gamma^\beta_{kl} S_{li}(z,w) \times \gamma^\alpha_{ij} S_{jm}(w,x) \tag{5.6j}$$

$$- ig^3\int d^4w d^4z\, \tilde{A}_\alpha(w)\{(\partial/\partial z^\beta)F(z,x)\} S_{nk}(y,z)\gamma^\beta_{kl} S_{li}(z,w) \times \gamma^\alpha_{ij} S_{jm}(w,x) \tag{5.6k}$$

$$- ig^3\int d^4w d^4z\, \tilde{A}_\beta(z)\{(\partial/\partial w^\alpha)F(w,x)\} S_{nk}(y,z)\gamma^\beta_{kl} S_{li}(z,w) \times \gamma^\alpha_{ij} S_{jm}(w,x) \tag{5.6l}$$

$$+ g^3\int d^4w\, F(y,x)\tilde{A}_\alpha(w) S_{ni}(y,w)\gamma^\alpha_{ij} S_{jm}(w,x) \tag{5.6m}$$

The diagrammatical representation of equation (5.6) is shown in Figure 6. The asterisk represents the fixed Coulomb source.

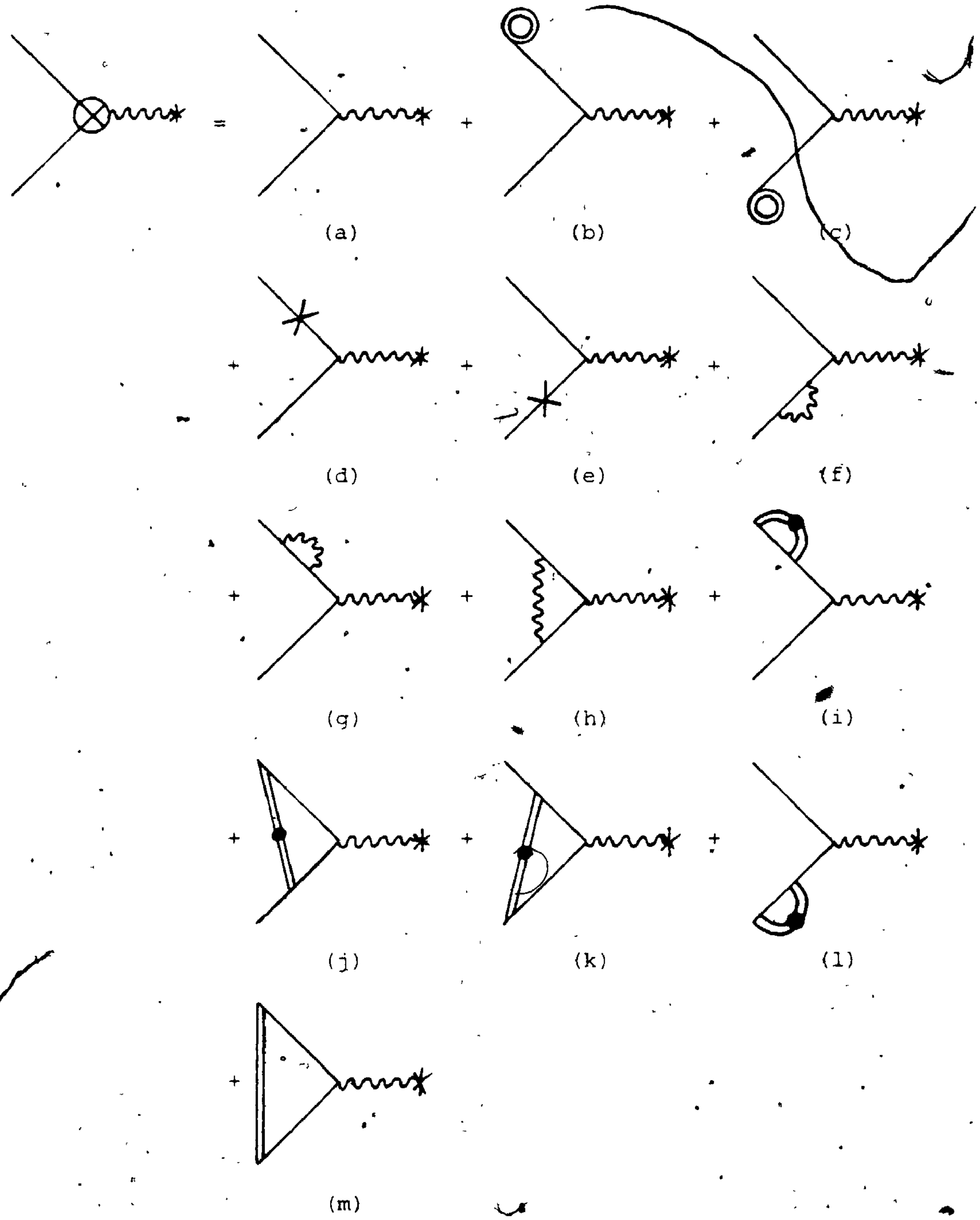


Figure 6. Diagrams contributing to the two point fermion field connected Green's function in the presence of an externally prescribed electromagnetic Coulomb potential to one loop order.

Again, making use of the integrals given in equation (4.14), the divergent contributions to the two point Green's function defined in equation (5.6) are:

$$(5.6a) = 0$$

$$(5.6b) = A_{mn}(x,y)$$

$$(5.6c) = A_{mn}(x,y)$$

$$(5.6d) + (5.6g) = A_{mn}(x,y)$$

$$(5.6e) + (5.6f) = A_{mn}(x,y)$$

$$(5.6h) = -A_{mn}(x,y)$$

$$(5.6i) = -A_{mn}(x,y)$$

$$(5.6j) = 0$$

$$(5.6k) = 0$$

$$(5.6l) = -A_{mn}(x,y)$$

$$(5.6m) = -A_{mn}(x,y)$$

where the function $A_{mn}(x,y)$ is defined according to the equation,

$$A_{mn}(x,y) \equiv -ig^3 \int d^4w\, \tilde{A}_\alpha(w) S_{ni}(y,w) \gamma^\alpha_{ij} S_{jm}(w,x) \{1/8\pi^2\} \ln(\Lambda/\mu) \qquad (5.7)$$

To one loop order the connected Green's function, and hence the S matrix, for Rutherford scattering from a fixed Coulomb potential with no bremsstrahlung emission is free of divergences, without even having to introduce a bilinear gauge or fermion field counterterm into the Lagrangian field density which breaks the gauge symmetry of the original theory. The only counterterm that is needed is the simple gauge invariant fermion field mass counterterm, the value of which is given in equation (4.21).

For the sake of comparison, consider the standard calculation of the two point fermion field connected Green's function in the presence of an externally prescribed electromagnetic Coulomb potential, as represented diagrammatically in Figure 7.

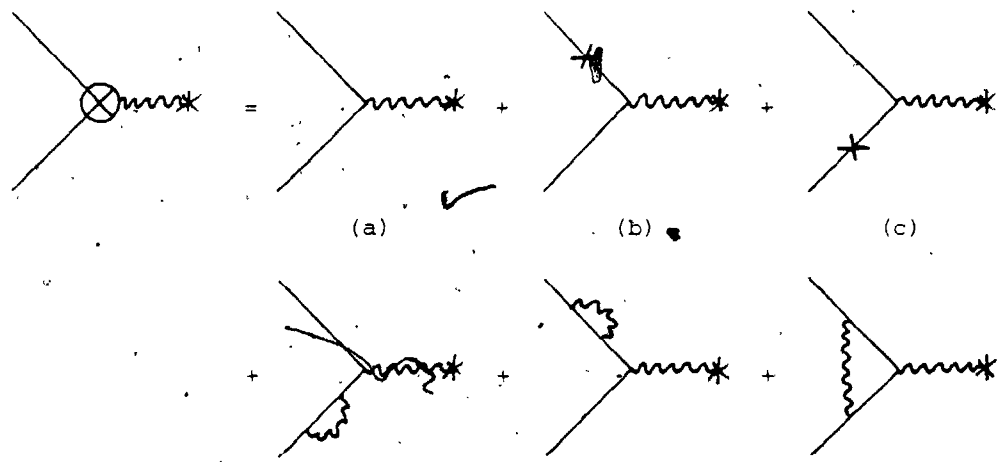


Figure 7. Diagrams contributing to the standard one loop order calculation of the two point fermion field connected Green's function in the presence of an externally prescribed electromagnetic Coulomb potential.

In complete analogy with the connected Green's function defined in equation (5.6), the connected Green's function in the standard formalism has the explicit form:

$$\langle 0|T(\bar{\phi}_m(x)\phi_n(y);\tilde{A}_\alpha(w))|0\rangle_c^{\text{standard}}$$
$$= (5.6a) + (5.6d) + (5.6e) + (5.6f) + (5.6g) + (5.6h) \qquad (5.8)$$

The divergent term in the connected Green's function defined in equation (5.8), when the Feynman gauge is used, has the value:

$$\mathrm{div}\{\langle 0|T(\bar{\psi}_m(x)\psi_n(y);\tilde{A}_\alpha(w))|0\rangle_c^{\text{standard}}\} = A_{mn}(x,y) \tag{5.9}$$

where $A_{mn}(x,y)$ is defined in equation (5.7).

The S matrix obtained from the Green's function defined in equation (5.8) will only be finite and physically realistic if a divergent fermion field wavefunction renormalization is performed.

In complete analogy with what is done in the standard formalism, the "infrared catastrophe", which is associated with the differential cross section for the emission of a very soft bremsstrahlung photon by an electron in the presence of a static Coulomb field, as given in equation (7.59) of Bjorken and Drell ( 7), can be avoided by replacing the two point gauge field Green's function for propagation on internal lines, $ig_{\mu\nu}G(x,y)$, with the modified two point Green's function, $ig_{\mu\nu}G^{(\lambda)}(x,y)$, where the Green's function $G^{(\lambda)}(x,y)$ has the form:

$$G^{(\lambda)}(x,y) = G^{(\lambda)}(x-y) = \int d^4p\, e^{-ip\cdot(x-y)}\{-1/(p^2-\lambda^2)\} \tag{5.10}$$

The presence of the small, fictitious mass, $\lambda$, the inclusion of which was first considered by Bloch and Nordsieck (12), has the effect of parametrizing the divergent contribution to the associated cross section from extremely low energy virtual photons on internal lines. When the limit as $\lambda$ tends to zero is taken the divergences that are present in the respective elastic and inelastic scattering cross sections cancel each other.

The inclusion of the mass parameter, $\lambda$, in the two point gauge field Green's function for propagation on internal lines, as given in equation (5.10), does not change any of the divergent terms in the integrals given in equation (4.14) since, it is only the Green's function, $F(x,y)$, with highly divergent polynomial behaviour near the origin of momentum space that contributes true infrared divergences, when regulated by the use of momentum cut-offs.

As a consequence of the fact that there is no such thing as free $\Lambda$ radiation, there is no possibility that a virtual $\Lambda$ 'particle', for momentum values very near, or at the pole of the propagator, can escape as free radiation from an internal line in any of the diagrams shown in Figure 6. If there is no amplitude for the emission of a physical $\Lambda$ 'particle', analogous to the amplitude for the emission of bremsstrahlung photons as given in equation (7.59) of Bjorken and Drell ( 7), then there is no associated "infrared catastrophe", and thus absolutely no physical basis to include a mass parameter, $\lambda$, in the Green's function, $F(x,y)$. It is also worth pointing out that since the $\Lambda$ 'particle' is related to the unphysical, longitudinal polarizations of the photon, that even if there were an "infrared catastrophe" associated with its emission, it would necessarily not be a divergence of a physical nature.

The preceding arguments, as to why there cannot be a mass parameter in the Green's function $F(x,y)$ are important, since the inclusion of one would render the contributions of diagrams like those shown in Figures 6(b), 6(c), 6(i), 6(l) and 6(m) finite, thereby affecting the precise cancellation between the ultraviolet divergences in the diagrams that are already present, and the divergences associated with diagrams involving this Green's function, when the total connected Green's function, as defined in equation (5.6), is calculated.

As alluded to earlier, the S matrix elements associated with a general n-point connected Green's function, for example those defined in equations (4.15), (4.18), (5.1) and (5.6), are obtained in the usual fashion by multiplying the respective connected Green's functions by the corresponding source functions and integrating over their respective spacetime variables. Thus, for the three point connected Green's function defined in equation (5.1), the associated S matrix element is:

$$S_{(3)} \equiv \int d^4x d^4y d^4v \left\{ \{iJ_1^\mu(v)\}\{-iK_{1m}(x)\}\{-i\overline{K}_{1n}(y)\} \times \langle 0|T(\overline{\phi}_m(x)\phi_n(y)B_\mu(v))|0\rangle_c \right\} \quad (5.11)$$

In order to eliminate the explicit dependence of a general S matrix element, such as $S_{(3)}$ defined above, on the source functions $K_1(x)$, $\overline{K}_1(x)$ and $J_1^\mu(x)$, consider the terms in the original Lagrangian field density which are independent of the coupling parameter, g. From equation (3.6), neglecting the term that fixes the gauge, which is inserted solely for the purpose of quantization, and neglecting the terms which contain the four source functions $\mathcal{J}(x)$, $J_2^\mu(x)$, $K_2(x)$ and $\overline{K}_2(x)$, since these only generate fields on internal lines, the action obtained from the Lagrangian field density of interest is:

$$S = \int d^4x \left\{ -(1/4)(\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x))(\partial^\mu A^\nu(x)-\partial^\nu A^\mu(x)) + i\overline{\psi}(x)\not{\partial}_x\psi(x) - m\overline{\psi}(x)\psi(x) + J_1^\mu(x)(A_\mu(x)+\partial_\mu\Lambda(x)) + \overline{K}_1(x)\psi(x) + \overline{\psi}(x)K_1(x) \right\} \quad (5.12)$$

Independent variation of this action with respect to the field variables $A_\mu(x)$, $\Lambda(x)$, $\overline{\psi}_i(x)$ and $\psi_j(x)$ yields the following Euler-Lagrange equations:

$$\{i\not\partial_x - m\}_{ij}\psi_j(x) + K_{1i}(x) = 0 \tag{5.13a}$$

$$\overline{\psi}_j(x)\{-i\overleftarrow{\not\partial}_x - m\}_{ji} + \overline{K}_{1i}(x) = 0 \tag{5.13b}$$

$$-\partial_\mu J_1^\mu(x) = 0 \tag{5.13c}$$

$$\{g^{\mu\nu}g_{\alpha\beta}\partial_x^\alpha\partial_x^\beta - \partial_x^\mu\partial_x^\nu\}A_\nu(x) + J_1^\mu(x) = 0 \tag{5.13d}$$

Note that equation (5.13c) can be obtained by taking the divergence of equation (5.13d). In this sense equation (5.13c) is redundant.

In terms of the Green's function defined in equation (3.12c), the solutions to equations (5.13a) and (5.13b) are:

$$\psi_i(x) = -\int d^4y\; S_{ij}(x,y)K_{1j}(y) \tag{5.14a}$$

$$\overline{\psi}_i(x) = -\int d^4y\; \overline{K}_{1j}(y)S_{ji}(y,x) \tag{5.14b}$$

In light of the discussion that was presented in the Introduction, with regard to inverting the operator $\Delta_x^{\mu\nu}$, as defined in equation (1.7), the solution to equation (5.13d) is:

$$A_\mu(x) = -\int d^4y\; G(x,y)g_{\mu\nu}J_1^\nu(y) \tag{5.15}$$

Rewriting the Green's function in the integrand of equation (5.15) in terms of the generalized function $D_{\mu\nu}(x,y)$, as defined in equation (3.19), leads to the equation:

$$A_\mu(x) = -\int d^4y\; D_{\mu\nu}(x,y)J_1^\nu(y) + \int d^4y\; \{(\partial/\partial x^\mu)(\partial/\partial y^\nu)F(x,y)\}J_1^\nu(y) \tag{5.16}$$

Integration by parts once, on the second term in equation (5.16), produces a factor of $\{(\partial/\partial y^\nu)J_1^\nu(y)\}$ which, to zeroth order in g, is identically zero. For the purposes of this work, a more useful solution to equation (5.13d) than the one given in equation (5.15) is:

$$A_\mu(x) = -\int d^4y\; D_{\mu\nu}(x,y)J_1^\nu(y) \tag{5.17}$$

This solution has already been quoted in equation (5.4), where it was used for the study of Rutherford scattering of an electron (or positron) off a fixed Coulomb potential.

The S matrix element associated with the lowest order contribution to the connected Green's function given in equation (5.6) is defined, in direct analogy with the S matrix defined in equation (5.11), by the equation:

$$S^{0}_{(2)}\{\tilde{A}_{\alpha}(w)\} = \int d^4x d^4y \left\{\{-iK_{lm}(x)\}\{-i\bar{K}_{ln}(y)\} ig\int d^4w\, \tilde{A}_{\alpha}(w) S_{ni}(y,w) \times \gamma^{\alpha}_{ij} S_{jm}(w,x)\right\} \quad (5.18)$$

Anticommuting the classical Grassman variables, $K_{lm}(x)$ and $\bar{K}_{ln}(y)$, and making use of equation (5.14) leads to the following result:

$$S^{0}_{(2)}\{\tilde{A}_{\alpha}(w)\} = ig\int d^4w\, \{\bar{\psi}_f(w)\tilde{\not{A}}(w)\psi_i(w)\} \quad (5.19)$$

The S matrix element given in equation (5.19) is exactly the same as the one given in equation (7.1) of Bjorken and Drell (7), when the relative minus sign between the two coupling parameters is taken into account. The indices f and i in equation (5.19) refer to the final and initial state wavefunctions, respectively.

The next to lowest order terms in the differential cross section for Rutherford scattering of an electron (or positron) in an external Coulomb potential can now be calculated following the procedure outlined in Sections 7.1 and 7.2 of Bjorken and Drell (7). The result that is obtained has no divergences associated with integration over internal loop momenta.

For the purpose of verifying the quantum electrodynamics vertex Ward Identity, it is necessary, as was done for the two point fermion field connected Green's function, to rearrange all of the terms given in equation (5.1) that contribute to the fermion-fermion-gauge field connected Green's function such that only a single fermion or gauge field propagator leads into each of the external spacetime points; x, y and v. The following results are obtained for all of those terms in equation (5.1) which involve more than one propagator ending on such an external spacetime point:

$$(5.1h) = -(5.1g) + 2g^3\int d^4w\,d^4z\,\Big\{S_{ni}(y,w)\gamma^{\rho}_{ij}S_{jm}(w,x)D^{\alpha}{}_{\mu}(z,v) \times \Big\{\Gamma_{\alpha\sigma}(\partial/\partial z^{\alpha})\{(\partial/\partial w^{\rho})(\partial/\partial z_{\sigma})F(z,w)\}\Big\}\Big\}. \qquad (5.20a)$$

$$(5.1n) = -ig^3\int d^4w\,d^4z\,d^4u\,\Big\{D_{\alpha\mu}(w,v)S_{nt}(y,u)\gamma^{\nu}_{tq}S_{qk}(u,z)\gamma^{\beta}_{kl} \times S_{li}(z,w)\gamma^{\alpha}_{ij}S_{jm}(w,x)\{(\partial/\partial u^{\nu})(\partial/\partial z^{\beta})F(z,u)\}\Big\} \qquad (5.20b)$$

$$(5.1o) = -g^3\int d^4w\,d^4z\,\Big\{D_{\beta\mu}(z,v)S_{nt}(y,z)\gamma^{\beta}_{tl}S_{li}(z,w)\gamma^{\alpha}_{ij} \times S_{jm}(w,x)\{(\partial/\partial w^{\alpha})F(w,z)\}\Big\}$$
$$- ig^3\int d^4w\,d^4z\,d^4u\,\Big\{D_{\beta\mu}(z,v)S_{nt}(y,u)\gamma^{\nu}_{tq}S_{qk}(u,z)\gamma^{\beta}_{kl} \times S_{li}(z,w)\gamma^{\alpha}_{ij}S_{jm}(w,x)\{(\partial/\partial u^{\nu})(\partial/\partial w^{\alpha})F(w,u)\}\Big\} \qquad (5.20c)$$

$$(5.1p) = -(5.1m) + ig^3\int d^4w\,d^4z\,d^4u\,\Big\{D_{\alpha\mu}(w,v)\gamma^{\alpha}_{ij}S_{jk}(w,z)\gamma^{\beta}_{kl}S_{li}(z,w) \times S_{nq}(y,u)\gamma^{\nu}_{qt}S_{tm}(u,x)\{(\partial/\partial u^{\nu})(\partial/\partial z^{\beta})F(z,u)\}\Big\} \qquad (5.20d)$$

$$(5.1q) = g^3\int d^4w\,d^4z\,\Big\{D_{\alpha\mu}(w,v)S_{nk}(y,z)\gamma^{\beta}_{kl}S_{li}(z,w)\gamma^{\alpha}_{ij} \times S_{jm}(w,x)\{(\partial/\partial z^{\beta})F(z,w)\}\Big\}$$
$$- ig^3\int d^4w\,d^4z\,d^4u\,\Big\{D_{\alpha\mu}(w,v)S_{nk}(y,z)\gamma^{\beta}_{kl}S_{li}(z,w)\gamma^{\alpha}_{ij} \times S_{jq}(w,u)\gamma^{\nu}_{qt}S_{tm}(u,x)\{(\partial/\partial u^{\nu})(\partial/\partial z^{\beta})F(z,u)\}\Big\} \qquad (5.20e)$$

$$(5.1r) = -ig^3\int d^4w\,d^4z\,d^4u \left\{D_{\beta\mu}(z,v)S_{nk}(y,z)\gamma^{\beta}_{kl}S_{li}(z,w)\gamma^{\alpha}_{ij} \times S_{jq}(w,u)\gamma^{\rho}_{qt}S_{tm}(u,x)\{(\partial/\partial u^{\rho})(\partial/\partial w^{\alpha})F(w,u)\}\right\} \tag{5.20f}$$

$$\begin{aligned}(5.1s) = {} & ig^3F(u,u)\int d^4w\,\{D_{\alpha\mu}(w,v)S_{ni}(y,w)\gamma^{\alpha}_{ij}S_{jm}(w,x)\} \\ & + g^3\int d^4w\,d^4z \left\{D_{\beta\mu}(z,v)S_{nt}(y,z)\gamma^{\beta}_{tl}S_{li}(z,w)\gamma^{\alpha}_{ij} \times S_{jm}(w,x)\{(\partial/\partial w^{\alpha})F(w,z)\}\right\} \\ & - g^3\int d^4w\,d^4z \left\{D_{\alpha\mu}(w,v)S_{nk}(y,z)\gamma^{\beta}_{kl}S_{li}(z,w)\gamma^{\alpha}_{ij} \times S_{jm}(w,x)\{(\partial/\partial z^{\beta})F(z,w)\}\right\} \\ & + ig^3\int d^4w\,d^4z\,d^4u \left\{D_{\alpha\mu}(w,v)S_{nk}(y,z)\gamma^{\rho}_{kq}S_{qi}(z,w)\gamma^{\alpha}_{ij} \times S_{jt}(w,u)\gamma^{\sigma}_{tl}S_{lm}(u,x)\{(\partial/\partial u^{\sigma})(\partial/\partial z^{\rho})F(z,u)\}\right\}\end{aligned} \tag{5.20g}$$

Collecting all of those terms in equation (5.1) which contain the generalized function $F(x,y)$ yields the following result:

$$\begin{aligned} & (5.1b) + (5.1c) + (5.1g) + (5.1h) + (5.1m) + (5.1n) + (5.1o) \\ & + (5.1p) + (5.1q) + (5.1r) + (5.1s) \\ & = 2g^3\int d^4w\,d^4z \left\{S_{ni}(y,w)\gamma^{\rho}_{ij}S_{jm}(w,x)D^{\alpha}{}_{\mu}(z,v) \times \left\{\Gamma_{\alpha\sigma}(\partial/\partial z^{\lambda})\{(\partial/\partial w^{\rho})(\partial/\partial z_{\sigma})F(z,w)\}\right\}\right\} \\ & - ig^3\int d^4w\,d^4z\,d^4u \left\{D_{\alpha\mu}(w,v)S_{nt}(y,u)\gamma^{\rho}_{tq}S_{qk}(u,z)\gamma^{\beta}_{kl} \times S_{li}(z,w)\gamma^{\alpha}_{ij}S_{jm}(w,x)\{(\partial/\partial u^{\rho})(\partial/\partial z^{\beta})F(z,u)\}\right\} \\ & - ig^3\int d^4w\,d^4z\,d^4u \left\{D_{\beta\mu}(z,v)S_{nt}(y,u)\gamma^{\rho}_{tq}S_{qk}(u,z)\gamma^{\beta}_{kl} \times S_{li}(z,w)\gamma^{\alpha}_{ij}S_{jm}(w,x)\{(\partial/\partial u^{\rho})(\partial/\partial w^{\alpha})F(w,u)\}\right\} \\ & + ig^3\int d^4w\,d^4z\,d^4u \left\{D_{\alpha\mu}(w,v)\gamma^{\alpha}_{ij}S_{jk}(w,z)\gamma^{\beta}_{kl}S_{li}(z,w) \times S_{nq}(y,u)\gamma^{\rho}_{qt}S_{tm}(u,x)\{(\partial/\partial u^{\rho})(\partial/\partial z^{\beta})F(z,u)\}\right\} \\ & - ig^3\int d^4w\,d^4z\,d^4u \left\{D_{\beta\mu}(z,v)S_{nk}(y,z)\gamma^{\beta}_{kl}S_{li}(z,w)\gamma^{\alpha}_{ij} \times S_{jq}(w,u)\gamma^{\rho}_{qt}S_{tm}(u,x)\{(\partial/\partial u^{\rho})(\partial/\partial w^{\alpha})F(w,u)\}\right\}\end{aligned} \tag{5.21}$$

Using this result, the fermion-fermion-gauge field three point connected Green's function, the value of which was previously given in equation (5.1), has the form defined in equation (5.22)

$$\langle 0|T(\bar{\phi}_m(x)\phi_n(y)B_\mu(v))|0\rangle_c$$

$$= -g\int d^4w\, D_{\alpha\mu}(w,v)S_{ni}(y,w)\gamma^\alpha_{ij}S_{jm}(w,x) \tag{5.22a}$$

$$+ 2g^3\int d^4w d^4z\, S_{ni}(y,w)\gamma^\rho_{ij}S_{jm}(w,x)D^\alpha{}_\mu(z,v)$$
$$\times \{\Gamma_{\alpha\sigma}(\partial/\partial z^\lambda)D^\sigma{}_\rho(z,w)\} \tag{5.22b}$$

$$+ g^3\int d^4w d^4z\, D_{\alpha\mu}(w,v)S_{nk}(y,z)\{\Sigma_{kl}(\partial/\partial z^\lambda)S_{li}(z,w)\}$$
$$\times \gamma^\alpha_{ij}S_{jm}(w,x) \tag{5.22c}$$

$$+ g^3\int d^4w d^4z\, D_{\alpha\mu}(w,v)S_{ni}(y,w)\gamma^\alpha_{ij}S_{jk}(w,z)$$
$$\times \{\Sigma_{kl}(\partial/\partial z^\lambda)S_{lm}(z,x)\} \tag{5.22d}$$

$$- ig^3\int d^4w d^4z d^4u\, \{S_{nr}(y,u)\gamma^\rho_{rt}S_{tk}(u,z)\gamma^\beta_{kl}S_{li}(z,w)\gamma^\alpha_{ij}$$
$$\times S_{jm}(w,x)D_{\alpha\beta}(w,z)D_{\rho\mu}(u,v)\} \tag{5.22e}$$

$$- ig^3\int d^4w d^4z d^4u\, \{S_{nr}(y,u)\gamma^\sigma_{rt}S_{tk}(u,z)\gamma^\beta_{kl}S_{li}(z,w)\gamma^\alpha_{ij}$$
$$\times S_{jm}(w,x)D_{\beta\sigma}(z,u)D_{\alpha\mu}(w,v)\} \tag{5.22f}$$

$$- ig^3\int d^4w d^4z d^4u\, \{S_{nr}(y,u)\gamma^\sigma_{rt}S_{tk}(u,z)\gamma^\beta_{kl}S_{li}(z,w)\gamma^\alpha_{ij}$$
$$\times S_{jm}(w,x)D_{\sigma\alpha}(u,w)D_{\beta\mu}(z,v)\} \tag{5.22g}$$

$$+ ig^3\int d^4w d^4z d^4u\, \{S_{ni}(y,w)\gamma^\alpha_{ij}S_{jm}(w,x)\gamma^\beta_{kl}S_{lr}(z,u)\gamma^\sigma_{rt}$$
$$\times S_{tk}(u,z)D_{\alpha\beta}(w,z)D_{\sigma\mu}(u,v)\} \tag{5.22h}$$

Directly analogous to the result that was obtained for the fermion field two point connected Green's function, the three point connected Green's function defined in equation (5.22) is just the result that is obtained in the conventional quantization treatment when the singular limit as $\lambda$ tends to zero is taken. This result, as previously discussed, which corresponds to the case of the unitary gauge, is free of any divergences.

After converting the two contributions to their respective connected Green's functions, defined in equations (4.26c) and (5.22g), into momentum space, followed by the removal of one propagator for each external fermion or gauge field line, the relationship between the one-particle-irreducible graphs so obtained can be analyzed exactly as is done in chapter 8 section 6 of Bjorken and Drell ( 7) for the case of the Feynman gauge, $\alpha = 1$. The Q.E.D. vertex Ward Identity is thus shown to be satisfied by the relevant Green's functions that are calculated using the quantization formalism considered in this work.

CHAPTER 6 - EXTENSION OF THE TECHNIQUE TO THEORIES WITH NON-ABELIAN GAUGE SYMMETRY

The Lagrangian field density for a pure Yang-Mills theory, (15), which has the following form,

$$L_{YM} = -(1/4)F^{a}_{\mu\nu}F^{a\mu\nu} \tag{6.1a}$$

where

$$F^{a}_{\mu\nu} \equiv \partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu} + gf^{abc}A^{b}_{\mu}A^{c}_{\nu} \tag{6.1b}$$

is invariant under the local gauge transformation specified by the equation:

$$A^{a}_{\mu}(x) \rightarrow A'^{a}_{\mu}(x) = A^{a}_{\mu}(x) + D^{ab}_{\mu}(A)\Lambda^{b}(x) \tag{6.2}$$

where the covariant derivative, $D^{ab}_{\mu}(A)$, is defined in the following way:

$$D^{ab}_{\mu}(A) \equiv \partial_{\mu}\delta^{ab} + gf^{apb}A^{p}_{\mu} \tag{6.3}$$

For the study of a quantized field theory with a non-Abelian gauge symmetry it is necessary to include the factor $\Delta_{f}\{\vec{A}_{\mu}\}$, as introduced in equation (4.27), in the functional path integral, in order to obtain an S matrix that is unitary. This factor has been shown to be independent of the gauge transformation function in Chapter 4.

Making use of the fact that the term $\Delta_{f}\{A^{a}_{\mu}\}$, as introduced in equation (4.27), which, for this particular theory has the explicit value:

$$\Delta_f\{A^a_\mu\} = \det\{D^{ab}_\mu(A)\,\partial^\mu\} \qquad (6.4)$$

can be written in the form:

$$\Delta_f\{A^a_\mu\} = \int\{dc^a\}\{d\bar{c}^a\}\ \exp\left\{i\int d^4x\{\bar{c}^a(x)D^{ab}_\mu(A(x))\,\partial^\mu_x c^b(x)\}\right\} \qquad (6.5)$$

where $c^a$ and $\bar{c}^a$ are complex, anti-commuting Grassman variables, the generating functional for pure Yang-Mills theory has the form given in equation (6.6), in direct analogy with the generating functional for the Abelian gauge field theory, defined in equation (3.1).

$$\begin{aligned} Z\{J_1^{a\mu},J_2^{a\mu},J^a,\xi^a,\bar{\xi}^a\} \\ = \int\{dA^a_\mu\}\{dc^a\}\{d\bar{c}^a\}\{d\Lambda^a\}\ \delta(\partial_\mu A^{a\mu} + \partial^\mu D^{ab}_\mu(A)\Lambda^b) \\ \times \exp\Big\{i\int d^4x\ \{-(1/4)F^a_{\mu\nu}F^{a\mu\nu} + \bar{c}^a D^{ab}_\mu(A)\,\partial^\mu c^b \\ + \Delta L_{YM} + J_1^{a\mu}(A^a_\mu + D^{ab}_\mu(A)\Lambda^b) + J_2^{a\mu}A^a_\mu \\ + J^a\Lambda^a + \bar{\xi}^a c^a + \bar{c}^a\xi^a\}\Big\} \end{aligned} \qquad (6.6)$$

Making use of the particular representation for the Dirac delta functional given in equation (3.4), equation (6.6) takes the following form:

$$\begin{aligned} Z\{J_1^{a\mu},J_2^{a\mu},J^a,\xi^a,\bar{\xi}^a\} \\ = \lim_{\alpha\to 0} \kappa(\alpha)\int\{dA^a_\mu\}\{dc^a\}\{d\bar{c}^a\}\{d\Lambda^a\}\ \exp\Big\{i\int d^4x\ \{-(1/4)F^a_{\mu\nu}F^{a\mu\nu} \\ + \bar{c}^a D^{ab}_\mu(A)\,\partial^\mu c^b + \Delta L_{YM} \\ - (1/2\alpha)\{\partial_\mu A^{a\mu} + \partial^\mu D^{ab}_\mu(A)\Lambda^b\}^2 \\ + J_1^{a\mu}(A^a_\mu + D^{ab}_\mu(A)\Lambda^b) + J_2^{a\mu}A^a_\mu + J^a\Lambda^a \\ + \bar{\xi}^a c^a + \bar{c}^a\xi^a\}\Big\} \end{aligned} \qquad (6.7)$$

Separating all of the terms in the exponent in equation (6.7) which contain a factor of g, from those which do not contain a factor of g, the generating functional can be rewritten:

$$Z\{J_1^{a\mu}, J_2^{a\mu}, J^a, \xi^a, \bar{\xi}^a\}$$

$$= \lim_{\alpha\to 0} \kappa(\alpha) \int \{dA_\mu^a\}\{dc^a\}\{d\bar{c}^a\}\{d\Lambda^a\} \exp\{i\int d^4x\, (L_g + L_{g'})\} \tag{6.8}$$

where

$$L_{g'} = -(1/2)(\partial_\mu A_\nu^a)(\partial^\mu A^{a\nu}) + (1/2)(\partial_\mu A_\nu^a)(\partial^\nu A^{a\mu}) + \bar{c}^a g_{\alpha\beta}\partial^\alpha\partial^\beta \delta^{ab} c^b$$
$$- (1/2\alpha)(\partial_\mu A^{a\mu} + g_{\alpha\beta}\partial^\alpha\partial^\beta\Lambda^a)^2 + J_1^{a\mu}(A_\mu^a + \partial_\mu\Lambda^a) + J_2^{a\mu}A_\mu^a + J^a\Lambda^a$$
$$+ \bar{\xi}^a c^a + \bar{c}^a \xi^a \tag{6.9a}$$

and

$$L_g = -(\partial_\mu A_\nu^a) g f^{abc} A^{b\mu} A^{c\nu} - (1/4) g^2 f^{abc} f^{apq} A_\mu^b A_\nu^c A^{p\mu} A^{q\nu}$$
$$+ \bar{c}^a g f^{apb} A_\mu^p (\partial^\mu c^b) - (1/\alpha)(\partial_\mu A^{a\mu} + g_{\alpha\beta}\partial^\alpha\partial^\beta\Lambda^a) g f^{apb} (\partial_\sigma A^{p\sigma}\Lambda^b)$$
$$- (1/2\alpha) g^2 f^{apb} f^{aqc} (\partial_\lambda A^{p\lambda}\Lambda^b)(\partial_\sigma A^{q\sigma}\Lambda^c) + g f^{apb} J_1^{a\mu} A_\mu^p \Lambda^b + \Delta L_{YM} \tag{6.9b}$$

Taking into account the presence of internal symmetry indices on the field variables in a non-Abelian gauge field theory, the gauge field and gauge transformation function content of $L_{g'}$ defined in equation (6.9a) is identical to that defined in equation (3.6), for an Abelian field theory. The lowest order contribution to the generating functional from these field variables is very similar to that already obtained for massive quantum electrodynamics, as given in equations (3.10) and (3.11).

$$\int \{dA_\mu^a\}\{dc^a\}\{d\bar{c}^a\}\{d\Lambda^a\} \exp\{i\int d^4x\, L_{g'}\} = \hat{\kappa}(\alpha) \exp\{i\tau(\alpha)\} \tag{6.10a}$$

where

$$\tau(\alpha) = (1/2)\int d^4x d^4y \Big\{ J_1^{a\mu}(x)\Big\{(\alpha-1)[(\partial/\partial x^\mu)(\partial/\partial y^\nu)]F(x,y)$$
$$- g_{\mu\nu}G(x,y)\Big\} J_1^{a\nu}(y) - 2J_1^{a\mu}(x) D_{\mu\nu}(x,y) J_2^{a\nu}(y)$$
$$+ (\alpha+1) J^a(x) F(x,y) J^a(y) \;[illegible]\; J_2^{a\mu}(x) G(x,y) J^a_{2\mu}(y)$$
$$+ 2\alpha J_1^{a\mu}(x)[(\partial/\partial x^\mu)F(x,y)]J^a(y)$$
$$- 2J_2^{a\mu}(x)[(\partial/\partial x^\mu)F(x,y)]J^a(y)\Big\}$$

$$- \int d^4x d^4y \, \{\bar{\xi}^a(x) G(x,y) \xi^a(y)\} \tag{6.10b}$$

Using the technique of functional differentiation with respect to the source functions, the connected Green's functions in this theory can be obtained in exactly the same manner as was done for the Abelian theory. The form of $(L_g)$, however, is considerably more involved, not only due to an increase in the actual number of perturbative terms, but also due to the fact that there will be terms, in the perturbation expansion of $\exp\{i\int d^4x \, L_g\}$, which explicitly contain the gauge parameter, $\alpha$, and which, when the limit as $\alpha$ tends to zero is taken, will introduce complications unless, to each and every order in g, for a given inverse power of $\alpha$, the sum of all such terms in a given connected Green's function is zero. The simplest such case to consider, for the purpose of illustration, is the contribution to the two point connected gauge field Green's function which contains two powers of g in the numerator and two powers of $\alpha$ in the denominator.

In the expansion of the factor $\exp\{i\int d^4x \, L_g\}$, in the generating functional, the only terms that are needed are:

$$\begin{aligned} W_{(2)} \equiv - (g^2/2\alpha^2) f^{ijk} f^{lmn} \int d^4w d^4z \Big\{ & \{(\partial/\partial w^\sigma)(\partial/\partial w^\lambda) A^{i\lambda}(w) \\ & + g_{\alpha\beta} \partial_w^\alpha \partial_w^\beta (\partial/\partial w^\sigma) \Lambda^i(w)\} A^{j\sigma}(w) \Lambda^k(w) \\ & \times \{(\partial/\partial z^\rho)(\partial/\partial z^\alpha) A^{l\alpha}(z) + g_{\xi\zeta} \partial_z^\xi \partial_z^\zeta (\partial/\partial z^\rho) \Lambda^l(z)\} \\ & \times A^{m\rho}(z) \Lambda^n(z) \Big\} \end{aligned} \tag{6.11}$$

The quantity $W_{(2)}$ defined in equation (6.11) is a valid simplification only if all of those configurations involving two point Green's functions which vanish in the limit as $\alpha$ tends to zero are ignored.

It is necessary, then, to calculate the following quantity:

$$\Pi^{ab}_{\mu\nu}(x,y) \equiv \langle 0|T(\beta^a_\mu(x)\beta^b_\nu(y)W_{(2)})|0\rangle_c \qquad (6.12)$$

where, in light of the discussion given before equation (6.12), the external field $\beta^a_\mu(x)$ cannot connect with the gauge transformation function $\Lambda^b(w)$.

The contribution to $\Pi^{ab}_{\mu\nu}(x,y)$, defined in equation (6.12), from the product of the first terms in the curly brackets in equation (6.11) has the explicit value:

$$\begin{aligned}\Pi 1^{ab}_{\mu\nu}(x,y) = &- (g^2/\alpha^2)C_2\delta^{ab}\int d^4w d^4z\, D_\mu{}^\rho(x,z)D_\nu{}^\sigma(y,w)\\ &\times\Big\{\{(\partial/\partial w^\sigma)(\partial/\partial z^\rho)F(w,z)\}\delta^4(w-z)\\ &- \{(\partial/\partial w^\sigma)G(w,z)\}\{(\partial/\partial z^\rho)G(z,w)\}\Big\}\end{aligned} \qquad (6.13a)$$

where

$$C_2\delta^{ab} \equiv f^{aik}f^{bik} \qquad (6.13b)$$

The fact that the Green's function, $D_\mu{}^\rho(x,z)$, is explicitly transverse, has been extensively used. Similarly, the cross terms from inside the curly brackets in equation (6.11) contribute a factor:

$$\begin{aligned}\Pi 2^{ab}_{\mu\nu}(x,y) = &\ (2g^2/\alpha^2)C_2\delta^{ab}\int d^4w d^4z\, D_\mu{}^\rho(x,z)D_\nu{}^\sigma(y,w)\\ &\times\Big\{\{(\partial/\partial w^\sigma)(\partial/\partial z^\rho)F(w,z)\}\delta^4(w-z)\\ &- \{(\partial/\partial w^\sigma)G(w,z)\}\{(\partial/\partial z^\rho)G(z,w)\}\Big\}\end{aligned} \qquad (6.14)$$

and finally, the product of the last terms in the curly brackets in equation (6.11) contributes a factor

$$\begin{aligned}\Pi 3^{ab}_{\mu\nu}(x,y) = &- (g^2/\alpha^2)C_2\delta^{ab}\int d^4w d^4z\, D_\mu{}^\rho(x,z)D_\nu{}^\sigma(y,w)\\ &\times\Big\{\{(\partial/\partial w^\sigma)(\partial/\partial z^\rho)F(w,z)\}\delta^4(w-z)\\ &- \{(\partial/\partial w^\sigma)G(w,z)\}\{(\partial/\partial z^\rho)G(z,w)\}\Big\}\end{aligned} \qquad (6.15)$$

A diagrammatic representation of the contributions to the one loop vacuum polarization tensor that are defined in equations (6.13), (6.14) and (6.15) is given in Figure 8.

(6.13) (6.14) (6.15)

Figure 8. Diagrammatic representation of the one loop contributions to the two point connected gauge field Green's function defined in equations (6.13), (6.14) and (6.15), respectively.

The total contribution to the two point connected Green's function defined in equation (6.12) is thus zero. There is no problem associated with taking the limit as $\alpha$ goes to zero in the term with this $\alpha$ and $g$ behaviour in the Green's function under consideration. As stated earlier, in order that there be no problem associated with taking the limit as $\alpha$ tends to zero in all terms which contribute to this connected Green's function, the coefficients of the $g^2/\alpha$, $g^4/\alpha^4$, $g^4/\alpha^3$, $g^4/\alpha^2$, $g^4/\alpha$, ... terms in the expansion of this Green's function must also all vanish identically and independently.

As considered in the context of pure Yang-Mills theory, the quantization technique developed in this thesis has been extended to the study of gauge field theories which exhibit invariance under the action of an internal symmetry group. The presence of the coupling parameter, $g$, in the gauge fixing term in the Lagrangian field density for theories such as this imposes, for the sake of finiteness and consistency when the limit is taken to zero for the gauge parameter, $\alpha$, the restriction that the coefficients of all terms in a given connected Green's function which behave as an inverse power of x must vanish. This type of restriction has been considered, by explicit calculation, for a particular term in the perturbative expansion of the two point connected gauge field Green's function.

## CHAPTER 7 - CONCLUSIONS

In conclusion, the gauge transformation variable, $\Lambda(x)$, has been treated as a 'functional' degree of freedom at each spacetime point, in addition to the gauge and fermion field variables, by integrating over all of those configurations which satisfy the gauge fixing condition when applied to the gauge transformed field variable. To one loop order, at least, in the connected Green's functions that have been considered, there is only a vacuum polarization divergence. The fermion field mass counterterm, though divergent, upholds the gauge invariance of the theory explicitly and is thus of significantly lesser importance.

The total $n$-point connected Green's functions involve, in addition to the diagrams that are present in the standard treatment, contributions from diagrams which contain the two point Green's function, $F(x,y)$, associated with the gauge transformation function, $\Lambda(x)$. These diagrams contain divergences which result from integration near the origin of the transform space variable. By appealing to an, albeit, heuristic argument, as presented in Chapter

4, it is possible to map true infrared divergences, such as these, into the more conventional ultraviolet type of divergence. In this way both of these kinds of divergences can be regulated with the same cut-off parameter, $\Lambda$, where the limit as $\Lambda$ tends to infinity is implied. For the purposes of this work, simple Laurent series expansions are used to extract the divergences in these integrals.

The calculations presented in this work have all been done using the technique of momentum cut-offs to regulate the divergent integrals which occur. As discussed in Chapter 4, however, even though the actual treatment of infrared and ultraviolet divergences in the context of dimensional regularization is different from that which is done using cut-offs, as illustrated by Lee and Milgram (11), this technique, first introduced by 't Hooft and Veltman ( 6), can be utilized for the purpose of obtaining consistent results.

With an increase of about a factor of three in the number of diagrams that contribute, in this formalism, to a general n-point connected Green's function, a calculation of the finite contribution to any S matrix or scattering cross section is a laborious task.

The S matrix, and hence differential cross section, for the Rutherford scattering of an electron (or positron) off a fixed Coulomb potential has been calculated to one loop order. Since the associated connected Green's function, as defined in equation (5.6), involves no external, propagating photons, to one loop order, there is no vacuum polarization divergence. In fact, providing allowance is made for the existence of a simple gauge invariant fermion field mass counterterm, this S matrix is completely finite. There is no need for a divergent fermion field wavefunction renormalization.

As pointed out in Chapter 5, in the discussion regarding the formation of S matrix elements from a general connected Green's function, the gauge transformation function, $\Lambda(x)$, does not appear in physical quantities such as scattering cross sections, due to the fact that the free Maxwell Lagrangian density, $L_{Maxwell} = -(1/4)F_{\mu\nu}F^{\mu\nu}$, contains a divergenceless operator which, when acting on the gradient of any function, gives a contribution of zero. This is very important since, unlike the gauge transformation equation for fermion fields, there is no coupling constant, $g$, in the transformation equation for the gauge field. Independent of the presence of any matter fields, the Maxwell field Lagrangian displays local gauge invariance. Furthermore, there is no possibility that the gauge transformation function can appear in a physical quantity accompanying the fermion fields since, to zeroth order in the coupling constant, which corresponds to asymptotic states far enough forward and backward in spacetime, the gauge transformed fermion field wavefunctions obey the same free Dirac equation as do the untransformed wavefunctions.

For the purposes of these calculations, $\epsilon$ is strictly set equal to zero, since, as introduced originally in the choice of the form for the Dirac delta functional given in equation (1.17), the limit as $\epsilon$ tends to zero must be taken in the connected Green's functions.

As illustrated in Chapter [illegible], field theories exhibiting invariance under non-Abelian gauge transformations, such as pure Yang-Mills theories, are amenable to the quantization approach discussed in this work. The appropriateness of this analysis, when applied to theories such as these, relies on the important fact, as explicitly shown for a simple case in Chapter 6, that, to any order in the coupling constant, $g$, for a particular connected Green's function, the coefficients of all terms containing powers of $1/\mu$ vanish identically and independently. This condition being met, the limit as $\mu$ tends to zero may be taken in a consistent manner, as there will be no artificial singularities introduced by this limiting procedure.

## APPENDIX 1. PRE-REGULARIZATION FOR SUPERSYMMETRY

It is well known that when studying quantum electrodynamics there are certain obvious advantages to parametrizing the singularities present in the loop momentum integrals by the method of dimensional regularization. In contrast to the non-transverse one loop vacuum polarization tensor, as calculated by Bogoliubov and Shirkov ( 2), which contains a quadratic divergence normally associated with a non-physical photon mass, that is obtained when the singular integrals are regulated with the aid of the technique of momentum cut-offs, the use of dimensional regularization results in a transverse, gauge invariant result for the one loop vacuum polarization tensor ( 4). The apparent 'suitability' of one particular regulating technique over all others for a particular problem is not all that uncommon, as it also occurs in the study of numerous other theories.

As pointed out by 't Hooft and Veltman ( 1), in dimensional regularization it is not possible to generalize a strictly four dimensional object such as the tensor $\epsilon_{\mu\beta\gamma\delta}$ to n dimensional space. The compromise that was originally introduced by Siegel ( 3) to circumvent this particular difficulty is to employ the regulating technique of dimensional reduction in which all algebraic manipulations are performed in four dimensions. Thus, only the momentum variables occurring in the loop integrals are extended to n dimensional objects.

Very similar to this problem is the one that is encountered when dimensional regularization is used to parametrize infinities in supersymmetric field theories, since continuation of all of the algebraic objects to n dimensional space breaks the fundamental Fermi-Bose supersymmetry. Again, as is the case for the tensor $\varepsilon_{\alpha\beta\gamma\delta}$ in quantum electrodynamics, in order to maintain consistency, all algebraic, or in this case superalgebraic, operations must be performed in four spacetime dimensions. As a result of this kind of behaviour, the study of such things as W.T.S.T. identities and anomalies is affected by the regulating technique that is chosen.

Making use of the additive ambiguity of the momentum variable in a given loop integral, and making use of the fact that integrals which are more than logarithmically divergent in the ultraviolet region have finite surface term contributions when the integration variable is shifted, it is possible, in all of those cases except where there is an anomaly present, to satisfy the expected generalized Ward identity, without being forced to choose a specific regulating technique.

A study of this analysis, called 'Pre-Regularization', has been done in collaboration with Victor Elias, Robb Mann and Gerry McKeon. The results of this initial work are contained in the reprint at the end of this appendix.

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# PRE-REGULARIZATION FOR SUPERSYMMETRY ☆

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Received 1 August 1983
Revised manuscript received 14 September 1983

A new regularization procedure for field theories is introduced that respects Ward–Takahashi–Slavnov–Taylor (WTST) identities while allowing divergences to be parametrized in any manner desired. Called pre-regularization, the procedure is particularly well-suited for supersymmetric field theories since the super-WTST identities are explicitly preserved in perturbation theory.

The inapplicability of conventional dimensional regularization (CDR) for supersymmetric theories is quite well known: CDR violates supersymmetry (SUSY) within finite parts of one-loop Feynman diagrams as well as within pole parts of two-loop Feynman diagrams [1]. Incompatibility of CDR with SUSY has led to the development of an alternative regularization procedure, regularization by dimensional reduction (RDR), a modified version of CDR in which fields (and $\gamma$-matrices) are explicitly four dimensional while momenta are taken to be less-than-four dimensional [2]. The intent of RDR is to maintain SUSY while continuing to parametrize Feynman-integral divergences in a gauge invariant manner, as in CDR.

Unfortunately, recent work has shown RDR to be an inconsistent procedure [3], producing ambiguities in higher orders of perturbation theory. Despite such ambiguities, it has been argued that RDR still remains unambiguous and consistent with SUSY in lower orders of perturbation theory [4]. Such arguments appear no longer to be tenable. RDR calculations of the $\beta$-function within $N = 1, 2$ and 4 super-Yang–Mills (SYM) theories have been shown to be ambiguous in three-loop order [5]. Moreover, SUSY violations arising within RDR calculations have been shown to occur beyond only one- or two-loop order (depending on the vertex under consideration) in component field calculations [5]. Within a superfield approach [6], calculational inconsistencies within RDR have been shown to be impossible to eliminate, even if SUSY violations are allowed [4]. Finally, even one-loop order calculational ambiguities have been shown to occur within RDR in the VVA triangle graph [7] and in the vacuum polarization of two-dimensional quantum electrodynamics [8] (which can be used to obtain an axial anomaly as well); such ambiguities are associated with the utilization of RDR-allowed four- (or two-) dimensional $\gamma$-matrix identities prior to projection of less-than-four- (or two-) dimensional momenta on $\gamma$-matrix indices.

Inconsistencies between SUSY and continuous-dimensional regularization techniques ultimately arise from the (selective) analytic continuation of certain quantities to a number of dimensions different from the number of spacetime dimensions. Indeed, perturbative field theory is *not* continuous with respect to the dimensionality of

☆ Work supported in part by the Natural Sciences and Engineering Research Council of Canada and by the National Science Foundation under Grant No. PHY82-15249.

Feynman integrals [9], nor does such continuation respect SUSY. One advantage of manipulating Feynman integrals within exactly four dimensions is that surface terms which arise from parametrizable integration-variable ambiguities in more-than-logarithmically divergent Feynman integrals are appropriately accounted for. Such terms, which do not occur when the dimensionality of spacetime is continued away from four [9], can be used to enforce SUSY and conventional (i.e., gauge) invariances of the lagrangian *regardless of how one eventually chooses to parametrize infinities.* We define "pre-regularization" to be the procedure by which surface terms, arising from parametrizable variable-of-integration ambiguities in four- [or (for extended SUSY) any integer] dimensional Feynman integration, are utilized to preserve conventional and super Ward–Takahashi–Slavnov–Taylor (WTST) identities [10]. Pre-regularization uses variable-of-integration ambiguities within integer-dimensional loop integrals to rewrite them into a form in which divergences may be parametrized without violating super-WTST identities. SUSY-violating quantities cancel for appropriate choices of integration variable. At no point in the procedure is there a need to continue the number of dimensions of any quantity in a manner that violates SUSY. Thus, Fierz transformations may be consistently performed, the four-dimensional $\epsilon$-tensor is retained, and the Kahane algorithm [11] for $\gamma$-matrix traces is upheld.

In order to demonstrate how pre-regularization works, consider pre-regularization of the graphs leading to the VVA-triangle anomaly. In fig. 1, the momenta $s_1$ and $s_2$ parametrize well-known arbitrariness in the loop momentum within the four-dimensional Feynman integrals associated with each graph. Let us denote by $S_{\mu\rho\sigma}$ the sum of the graphs of fig. 1, where $\mu$ is understood to correspond to the axial-vector vertex. In four dimensions, divergences of vector and axial-vector triangle-graph currents are found to be proportional to the differences of linearly divergent Feynman integrals that are related to each other by shifts in the variable of integration. Such differences are proportional to finite surface terms that involve the as yet arbitrary parameters $s_1$ and $s_2$ [7]:

$$k_1^{\sigma} S_{\mu\rho\sigma} = (-ie^2/8\pi^2)\epsilon_{\mu\tau\rho\eta}[(-k_2 + s_2 - s_1)^{\tau} k_1^{\eta}] , \tag{1}$$

$$k_2^{\rho} S_{\mu\rho\sigma} = (ie^2/8\pi^2)\epsilon_{\mu\tau\sigma\eta}[(2k_1 + s_2 - s_1)^{\tau} k_2^{\eta}] , \tag{2}$$

$$(k_1 + k_2)^{\mu} S_{\mu\rho\sigma} = -2mP_{5\rho\sigma} - (ie^2/8\pi^2)\epsilon_{\rho\tau\sigma\eta}[(s_2 - s_1 - k_1)^{\tau}(k_1 + k_2)^{\eta}] . \tag{3}$$

$P_{5\rho\sigma}$ is just $S_{\mu\rho\sigma}$ with $\gamma_\mu\gamma_5$ replaced by $\gamma_5$ [12]. Since $s_1$ and $s_2$ are arbitrary, their difference may be expressed in terms of the external momenta $k_1$ and $k_2$:

$$(s_2 - s_1)^{\tau} = A(k_1, k_2)k_1^{\tau} + B(k_1, k_2)k_2^{\tau} \tag{4}$$

Pre-regularization of the graphs of fig. 1 is accomplished by insisting that $s_1$ and $s_2$ be chosen in a manner consis-

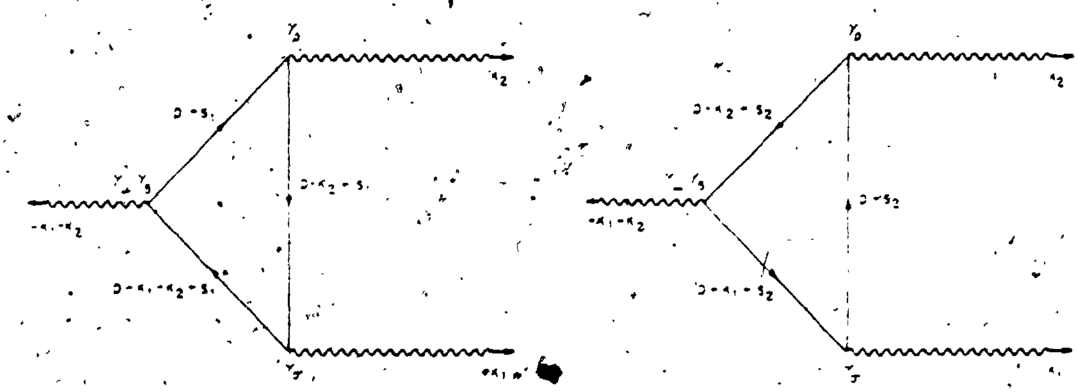

Fig. 1. VVA triangle graph and crossed graph.

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tent with vector-current conservation Ward identities. We would therefore require in eq. (4) that $B(k_1, k_2) = 1$ and that $A(k_1, k_2) = -2$ in order to eliminate the right-hand sides of eqs. (1) and (2). We see that satisfying vector-current conservation uniquely determines $s_2 - s_1$, further pre-regularization to satisfy naive axial-vector current Ward identities [consistent with having the right-hand side of eq. (3) equal $-2mP_{5\rho\sigma}$] is impossible. Indeed, our pre-regularization prescription leads to the usual anomalous divergence of the axial vector current:

$$(k_1 + k_2)^{\mu} S_{\mu\rho\sigma} = -2mP_{5\rho\sigma} + (ie^2/2\pi^2)\epsilon_{\rho\tau\sigma\eta} k_1^{\tau} k_2^{\eta} . \tag{5}$$

We stress that eqs. (1)–(3) are obtained using conventional (four-dimensional) $\gamma$-matrix algebra; it is the absence of such shift-of-integration-variable surface terms when Feynman integrals are continued away from four-dimensions [9] that entails the introduction of ad hoc $n$-dimensional $\gamma$-matrix rules in CDR [13] (in which $\gamma_5$ does not anticommute with all $\gamma_\mu$) and RDR [14] (in which only some $\gamma$-matrices retain cyclicity) in order to reproduce the axial anomaly. Also note that eq. (5) is obtained without reference to how perturbation-theory infinities are to be eventually parametrized.

To further illustrate pre-regularization in four dimensions, consider the Ward identity relating the fermion self-energy $\Sigma(p)$ (fig. 2a) to the vertex correction $\Lambda(p, p')$ (fig. 2b) in quantum electrodynamics:

$$-e\, \partial\Sigma(p)/\partial p_\lambda = \Lambda^{\lambda}(p, p) . \tag{6}$$

Arbitrariness in the loop momentum is denoted by the parameter $s$ in fig. 2. Pre-regularization of the self-energy entails finding a value for $s$ consistent with eq. (6). From fig. 2a, it is straightforward to show, in four dimensions, that

$$-e \frac{\partial}{\partial p_\lambda} \Sigma(p) = -2e^3 \frac{\partial}{\partial p_\lambda}\left( \int \frac{d^4k}{(2\pi)^4} (\tfrac{1}{2}\not{p} - \not{k} - \not{s}) \int_0^1 dx\, \{[k + s + \tfrac{1}{2}p(1-2x)]^2 + p^2 x(1-x)\}^{-2} \right) \tag{7}$$

If we shift the variable of integration from $k$ to $k + s + \frac{1}{2}p(1-2x)$, we obtain a surface term depending on the parameter $s$ [15]:

$$-e \frac{\partial}{\partial p_\lambda} \Sigma(p) = -2e^3 \frac{\partial}{\partial p_\lambda}\left( \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{[\not{p}(1-x) - \not{k}]}{[k^2 + p^2 x(1-x)]^2} - \frac{1}{32\pi^2} \int_0^1 dx\, [\not{s} - \tfrac{1}{2}\not{p}(1-2x)] \right)$$

$$= -2e^3 \left[ \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \left( \frac{(1-x)\gamma^{\lambda}}{[k^2 + x(1-x)p^2]^2} - \frac{4x(1-x)^2 p^{\lambda}\not{p}}{[k + x(1-x)p^2]^3} \right) - \frac{1}{32\pi^2} \frac{\partial \not{s}}{\partial p_\lambda} \right] \tag{8}$$

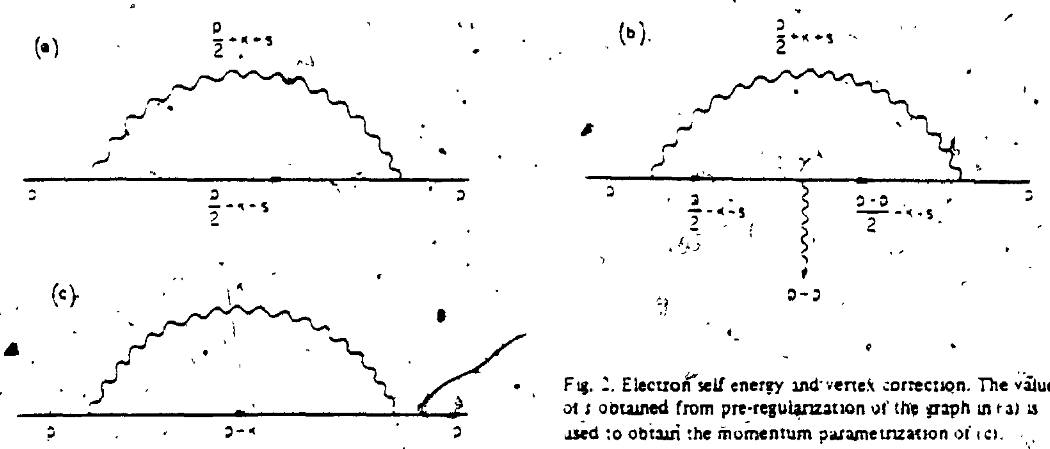


Fig. 2. Electron self energy and vertex correction. The value of $s$ obtained from pre-regularization of the graph in (a) is used to obtain the momentum parametrization of (c).

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Evaluation of $\Lambda^\lambda(p,p)$ from fig. 2b is insensitive to loop-momentum ambiguities, as $\Lambda^\lambda(p,p')$ is only log-divergent [15]. Using the four-dimensional $\gamma$-matrix identity $\gamma_\mu \not{a}\not{b}\not{c}\gamma^\mu = -2\not{c}\not{b}\not{a}$, we find that

$$\Lambda^\lambda(p,p) = -2e^3 \int \frac{\mathrm{d}^4k}{(2\pi)^4} \int_0^1 \mathrm{d}x\, \frac{[-xk^2\gamma^\lambda + 2x(1-x)^2p^2\gamma^\lambda - 4x(1-x)^2\not{p}p^\lambda]}{[k^2 + x(1-x)p^2]^3}\,. \tag{9}$$

Subtracting eq. (9) from eq. (8) gives

$$-e\frac{\partial}{\partial p_\lambda}\Sigma(p) - \Lambda^\lambda(p,p) = 2e^3\left(\int \frac{\mathrm{d}^4k}{(2\pi)^4}\int_0^1 \mathrm{d}x\, \frac{x(1-x)^2p^2\gamma^\lambda}{[k^2+x(1-x)p^2]^3} + \frac{1}{32\pi^2}\frac{\partial s}{\partial p}\right) = \frac{ie^3}{16\pi^2}\left(\tfrac{1}{2}\gamma^\lambda + \frac{\partial s}{\partial p_\lambda}\right) \tag{10}$$

Hence, the Ward identity is satisfied provided we choose $s = -p/2$, a pre-regularization consistent with fig. 2c, which is, in fact, the loop-momentum parametrization used by Jauch and Rohrlich [15] to derive eq. (6) from explicit surface terms of the self-energy in four dimensions.

Of course, eq. (6) is automatically satisfied in dimensional regularization. Had we chosen to integrate in $n$ dimensions and then applied the $n$-dimensional $\gamma$-matrix identities $\gamma_\mu \not{a}\gamma^\mu = (2-n)\not{a}$, $\gamma_\mu \not{a}\not{b}\not{c}\gamma^\mu = 2(4-n)[a\cdot b\not{c} - a\cdot c\not{b} + b\cdot c\not{a}] + (n-6)\not{c}\not{b}\not{a}$, we would have found that

$$-\frac{\partial}{\partial p^\lambda}\Sigma^{(n)}(p) - \Lambda^{(n)}_\lambda(p,p) = (2-n)e^3\left(-\int\frac{\mathrm{d}^nk}{(2\pi)^n}\int_0^1 \mathrm{d}x\, \frac{[(n-4)/n]\,xk^2\gamma_\lambda + x(1-x)^2p^2\gamma_\lambda}{[k^2+x(1-x)p^2]^3} - \delta_{n,4}\frac{1}{32\pi^2}\frac{\partial s}{\partial p_\lambda}\right) \tag{11}$$

If the integrals on the right-hand side of eq. (11) are evaluated for $n \neq 4$, the numerator factor of $(n-4)$ cancels a pole at $n = 4$ in the Feynman integral. The right-hand side of eq. (11) then vanishes if the $n \to 4$ limit is taken from above or below. [Note that $\mathrm{Lim}(\delta_{n\to 4})$ is *zero* rather than one; the discontinuity within $f(n) = \delta_{n,4}$ is manifested by inequality between $f(4)$ (=1) and the limiting value (zero) attained by $f(n)$ as $n$ approaches 4 from above or below.] The point we wish to make here is that one does not *need* to continue to $n$-dimensions to preserve standard Ward identities. Indeed, our pre-regularization condition that $s = -p/2$ utilises an arbitrary surface term (associated with a loop-momentum-variable ambiguity) occurring in (exactly) four dimensions to maintain a standard Ward identity *without invoking continuation to a dimensionality different from that of spacetime*. It is precisely this feature that is required to maintain super-WTST-identities in supersymmetric theories.

We now consider a simple example of pre-regularization in a supersymmetric theory by considering the vacuum polarization within SUSY quantum electrodynamics. Diagrams contributing to the vector superfield ($V$) vacuum polarization are listed in fig. 3. Arbitrariness in the definition of loop-momentum variables within figs. 3a and 3b is parametrized by $s_1$ and $s_2$, respectively. Upon clearing reducible products of super-covariant derivatives from internal lines (associated with chiral superfields), we find that the contribution of fig. 3a to the vacuum polarization of the vector superfield is given by [6]

$$\Gamma_a = g^2\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k+s_1)^2(p+k+s_1)^2}\int \mathrm{d}^4\theta\, V(-p,\theta)[-(k+s_1)^2 - \tfrac{1}{2}(k+s_1)^{\alpha\dot\beta}D_\alpha\bar{D}_{\dot\beta} + \tfrac{1}{16}D^2\bar{D}^2]V(p,\theta) \tag{12}$$

The contribution of fig. 3b (which vanishes under CDR) is given by

$$\Gamma_b = g^2\int\frac{\mathrm{d}^4p}{(2\pi)^4}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{(k+s_2)^2}\int \mathrm{d}^4\theta\, V(-p,\theta)V(p,\theta)\,. \tag{13}$$

The sum of $\Gamma_a$ and $\Gamma_b$ includes the difference of two quadratically divergent integrals, which in four dimensions yields a surface term involving $s_1$ and $s_2$ [16].

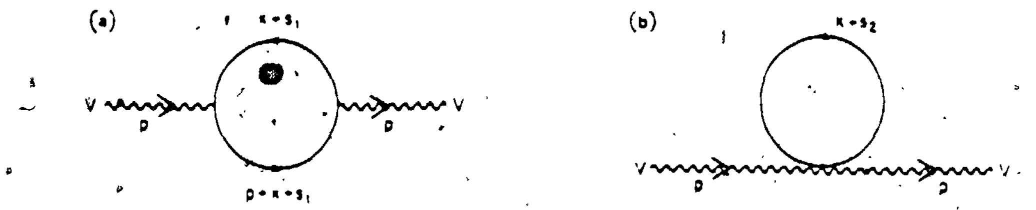


Fig. 3. Graphs contributing to the vector super-field vacuum polarization in supersymmetric QED. Solid lines are chiral superfields.

$$\int \frac{d^4k}{(p+k+s_1)^2} - \int \frac{d^4k}{(k+s_2)^2} = -\tfrac{1}{2} i\pi^2 [(p+s_1)^2 - s_2^2] . \tag{14}$$

There is also a linearly divergent term in eq. (12) [6] which yields a surface term dependent on $s_1$ [9,10]:

$$\int \frac{d^4k\,(k+s_1)^{\alpha\dot\beta}}{(k+s_1)^2(p+k+s_1)^2} = \int \frac{d^4k\,k^{\alpha\dot\beta}}{k^2(p+k)^2} + \tfrac{1}{2} i\pi^2 s_1^{\alpha\dot\beta} = -\tfrac{1}{2} p^{\alpha\dot\beta} \int \frac{d^4k}{k^2(p+k)^2} + \tfrac{1}{4} i\pi^2 [p^{\alpha\dot\beta} + 2s_1^{\alpha\dot\beta}] . \tag{15}$$

Substitution of (14) and (15) into the sum of (12) and (13) leads to the following expression for the vacuum polarization of the vector superfield.

$$\Gamma_a + \Gamma_b = g^2 \int \frac{d^4p}{(2\pi)^4} A(p) \int d^4\theta\, V(-p,\theta)(\tfrac{1}{2} p^{\alpha\dot\beta} D_\alpha \bar D_{\dot\beta} + \tfrac{1}{16} D^2 \bar D^2) V(p,\theta)$$

$$- \frac{i\pi^2 g^2}{8} \int \frac{d^4p}{(2\pi)^4} \int d^4\theta\, V(-p,\theta)[(p+s_1)^2 - 4s_2^2 + (p^{\alpha\dot\beta} + 2s_1^{\alpha\dot\beta}) D_\alpha \bar D_{\dot\beta}] V(p,\theta) , \tag{16}$$

where $A(p) \equiv (2\pi)^{-4} \int d^4k\,[k^2(p+k)^2]^{-1}$. Only the first term in square brackets may be absorbed by a superinvariant rescaling of the superfield lagrangian; the remaining terms preclude such a superinvariant rescaling and must be eliminated by choosing $s_1^\mu = -p^\mu/2 = \pm s_2^\mu$. At this point, one can parametrize the infinities within $A(p)$ any way one wishes (including as poles at $n = 4$), since explicit SUSY invariance has been respected through pre-regularization.

The examples listed above are consistent with the following general procedure for pre-regularizing a calculation:

(i) To a given order in perturbation theory, write down all Feynman integrals as $D$-dimensional integrals, where $D$ is the dimensionality of spacetime. Parametrize the arbitrariness in each independent loop momentum $k_i$ with an additional momentum-shift parameter $s_i$ (consistent with momentum conservation at each vertex). Each integral will be of the form

$$I = \prod_{i=1}^{n} \int d^D k_i\, F(k_i + s_i, p_j, m_j) , \tag{17}$$

where $p_j$ and $m_j$ are external momenta and particle masses. Note that $D$ is an integer; for $N = 1$ SUSY theories, $D = 4$.

(ii) Using the techniques developed in ref. [9] for shifting integration variables within integer-dimension Feynman integrals, rewrite (17) as follows:

$$I = \prod_{i=1}^{n} \prod_{l=1}^{n} \int d^D k_i \int_0^1 dx_l\, \tilde F(k_i^2; p_j, m_j, x_l) - S(s_i, p_j, m_j) . \tag{18}$$

where $x_i$ are Feynman parameters, the number $(n')$ of which is zero or finite. $\tilde{F}$ *by definition* contains no momentum shift parameters $s_i$; it is obtained from the function $F$ in (17) by shifts of integration variable such that the final result for $\tilde{F}$ depends only on squares of loop momenta [9]. All dependence on the shift parameters is contained in the function $S$; evaluation of $S$ may involve integrations over a subset of the loop momenta and/or Feynman parameters. The transition from eq. (7) to eq. (8) is an example of the transition from eq. (17) to eq. (18).

(iii) *Choose* $\{s_i^\mu\}$ such that $I$ satisfies the WTST identities of the theory. In general, each $s_i$ will be a linear combination of external momenta and particle masses:

$$s_i^\mu = \sum_j \alpha_{ij} p_j^\mu + \sum_l \beta_{il} m_l , \qquad (19)$$

the parameters $\alpha_{ij}$ and $\beta_{il}$ are (possibly divergent) functions of $\{p_j^\mu\}$ and $\{m_l\}$, the index $\tilde{}$ denotes the fact that different parametrizations of field-theoretic divergences may require different choices of $\alpha_{ij}$ and $\beta_{il}$ (such was not, however, the case in the examples presented here). Nevertheless the only constraint on these coefficients is that substitution of (19) into (18) will yield an expression invariant under WTST identities for whichever way we choose to parametrize any divergences remaining within (18). Note that the occurrence of anomalies within a given theory is manifested by having an insufficient number of coefficients $\alpha_{ij}$ and $\beta_{il}$ (or, alternatively, an insufficient number of independent $s_i$ parametrizing loop-momentum ambiguities) to constrain simultaneous preservation of all WTST identities. Such is precisely the case for the VVA triangle: two coefficients [eq. (4)] were not sufficient to guarantee Ward identities associated with conservation of currents at all three vertices.

(iv) Having performed these three steps, the expression obtained will equal the first term on the right-hand side of (18) $(\tilde{F})$ plus an additional term containing some subset of $\{\alpha_{ij}, \beta_{il}\}$. We denote any such coefficients which enter into the final expression for $I$ as "external shift coefficients" (ESCs), those coefficients which serve only to cancel WTST-identity violating pieces (also partly arising from surface terms) we will denote as "internal shift coefficients" (ISCs).

Unitarity implies that any remaining ESCs in a given diagram yield the same contribution on all possible Riemann sheets of external momentum invariants of $I$ [17]. In particular, ESCs are not permitted to contribute to the imaginary part of the scattering amplitude. The unitarity constraint is satisfied provided ESCs are meromorphic functions of external momentum invariants. Such ESCs behave like additional renormalizations which can be fixed by an appropriate choice of renormalization conditions. The divergences in the final expression may be parametrized in any manner desired.

To conclude, we reiterate that CDR and RDR manifestly preserve WTST identities in non-SUSY theories precisely because such identities do not depend on the dimensionality of Feynman integrals. Super-WTST identities, however, are inextricably linked to spacetime, rendering CDR inapplicable. RDR attempts to accommodate specificity of super-WTST identities to spacetime without abandoning the continuous-dimension parametrization of field-theoretic divergences; this approach is now known to be inconsistent [3,5]. In the pre-regularization procedure proposed here, all quantities are manipulated in the same dimensionality as spacetime. SUSY is therefore upheld, and WTST identities are preserved through utilization of the arbitrariness present in the definition of the loop momentum, manifested as surface terms within integer-dimension Feynman integrals.

We have applied pre-regularization to scalar QED, spinor-QED, Yang–Mills theories and scalar field theories, and have found that the problems created by abandoning CDR are in each case eliminated by an appropriate choice of momentum-shift parameters $s_i$. We have also used the method to reproduce the axial anomaly in two-dimensional QED. Those results will be presented in detail later [18]. Utilization of pre-regularization beyond one-loop order is currently under investigation as well; it has already been demonstrated that retention of shift-of-integration-variable surface terms within a spontaneously broken U(1) theory does not affect the cancellation of overlapping divergences in two-loop order [16].

We would like to thank R. and D. Mackenzie for helpful suggestions.

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## APPENDIX 2. LONGITUDINAL CONTRIBUTIONS TO THE VACUUM POLARIZATION IN THE 'T HOOFT-VELTMAN GAUGE

In quantized pure Maxwell theory when a non-linear gauge fixing condition is imposed on the functional path integral, the two point connected Green's function has a perturbative structure. An example of this kind of gauge fixing condition is the 't Hooft-Veltman gauge ( 1), where the following constraint applies:

$$\partial \cdot A + \lambda A^2 = 0 \tag{1}$$

The vacuum polarization tensor, for this choice of gauge fixing condition, is a perturbative quantity in the dimensionless parameter, $\lambda$. However, it is also non-transverse, a fact which requires the introduction of non-gauge invariant counterterms in the Lagrangian field density. In terms of the Ward-Takahashi-Slavnov-Taylor identities, this longitudinal term in the vacuum polarization tensor is compensated by the presence of 'pincer' type diagrams. The results of a study of this identity, carried out in collaboration with R. B. Mann and G. McKeon, are presented, in the form of a reprint, at the end of this appendix.

Further study of this theory, which has been done using the formalism of Becchi, Rouet and Stora ( 2), in collaboration with Gerry McKeon, Sanjiv Samant and Tom Sherry, from the National University of Ireland, University College, Galway, indicates that it is the longitudinal term in the vacuum polarization that preserves the B.R.S. invariance of this theory.

For the purpose of calculating the Feynman Path Integral, the full Lagrangian field density has the form:

$$\mathcal{L} = -(1/4)(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - (1/2)(\partial\cdot A + \lambda A^2)^2 + \bar{c}(\partial_\mu + 2\lambda A_\mu)\partial^\mu c \qquad (2)$$

where $c$ and $\bar{c}$ are the non-physical Faddeev-Popov fields. The B.R.S. transformations, which leave the Lagrangian field density given in equation (2) invariant, have the form:

$$\delta A_\mu = (\partial_\mu c)\varepsilon \qquad (3a)$$

$$\delta c = 0 \qquad (3b)$$

and

$$\delta\bar{c} = -(\partial_\mu A^\mu + \lambda A^2)\varepsilon \qquad (3c)$$

In equation (3) $\varepsilon$ is a constant, infinitesimal anti-commuting parameter.

The B.R.S. identity relating the vacuum polarization and the 'pincer' type terms is most easily studied by considering the simple two point connected Green's function defined in equation (4).

$$G_\mu(x,y) \equiv \langle 0|T(A_\mu(x)\bar{c}(y))|0\rangle_c \qquad (4)$$

In order for the B.R.S. invariance to be maintained, the connected Green's function, $G_\mu(x,y)$, defined in equation (4) must be invariant under the transformations of the field variables specified in equation (3). Removing the factor of $\varepsilon$, this implies

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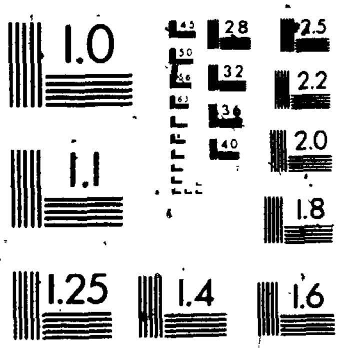

the following identity:

$$<0|T\{(\partial_\mu c(x))\bar{c}(y)\}|0>_c + <0|T\{A_\mu(x)(\partial^\nu A_\nu(y))\}|0>_c + \lambda<0|T\{A_\mu(x)A_\nu(y)A^\nu(y)\}|0>_c = 0 \quad (5)$$

Performing the explicit calculation of the terms in equation (5) using dimensional regularization to parametrize the infinities leads to the following results:

$$<0|T\{(\partial_\mu c(x))\bar{c}(y)\}|0>_c = \{2i\lambda^2/(2\pi)^8\}(p_\mu/p^2)I \quad (6a)$$

$$<0|T\{A_\mu(x)(\partial_\nu A^\nu(y))\}|0>_c = \{2i\lambda^2/(2\pi)^8\}(n-2)(p_\mu/p^2)I \quad (6b)$$

and

$$\lambda<0|T\{A_\mu(x)A_\nu(y)A^\nu(y)\}|0>_c = -\{2i\lambda^2/(2\pi)^8\}(n-1)(p_\mu/p^2)I \quad (6c)$$

In equation (6) the quantity I is defined in the following manner:

$$I \equiv \int d^nk\ \{1/(2\pi)^n\}\{1/(k^2(k+p)^2)\} \quad (7)$$

Clearly, the identity defined in equation (5) is satisfied. This would not be the case if the vacuum polarization tensor were transverse.

It has been shown, by explicit construction, that the presence of a longitudinal term in the vacuum polarization tensor is necessary for the satisfaction of both the W.T.S.T. and B.R.S. identities. As these identities involve Green's functions only, even taking into account the extra longitudinal terms in the vacuum polarization tensor, the S matrix elements that are calculated using this choice of gauge fixing term are consistent with those that are calculated using the more standard choices of gauge fixing terms.

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# Longitudinal contributions to the vacuum polarization in the 't Hooft–Veltman gauge

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Received May 14, 1984

The Ward–Takahashi–Slavnov–Taylor (WTST) identities are examined in the 't Hooft–Veltman gauge $\partial\cdot A - \lambda A^2 = 0$. It is found that gauge invariance implies that there is a contribution to the longitudinal part of the vacuum polarization that is proportional to $\lambda$.

On examine les identités Ward–Takahashi–Slavnov–Taylor (WTST) dans la jauge 't Hooft–Veltman $\partial\cdot A - \lambda A^2 = 0$. On trouve que l'invariance de jauge implique qu'il y a une contribution à la partie longitudinale de la polarisation du vide qui est proportionnelle à $\lambda$.

[Traduit par le journal]

Can. J. Phys. 62, 1129 (1984)

## Introduction

In their discussion of the WTST identities and gauge invariance in field theory, 't Hooft and Veltman (1) found that the gauge-fixing Lagrangian in electrodynamics.

$$[1]\quad \mathcal{L}_{gf} = -\tfrac{1}{2}(\partial\cdot A - \lambda A^2)^2$$

is useful in illustrating the role played by ghost particles.

The calculation of the vacuum polarization tensor in this gauge is complicated by having to take into account vertices induced by terms dependent on $\lambda$ in [1]. In ref. 1, 't Hooft and Veltman made the simplifying assumption that the photons couple to transverse external sources, and hence project out only those parts of the vacuum polarization tensor that are proportional to $g_{\mu\nu} - k_\mu k_\nu/k^2$. One finds, though, that an explicit calculation using a gauge invariant regularization procedure, such as dimensional regularization (2), shows that the vacuum polarization does receive a longitudinal contribution proportional to $\lambda$.

## Discussion

When one examines the WTST identities appropriate to the 't Hooft–Veltman gauge of [1], it is found that such longitudinal contributions are not unexpected. This is illustrated below.

Capper and Leibbrandt (3) have shown that something similar happens in the planar gauge, where

$$[2]\quad \mathcal{L}_{gf} = \frac{1}{2\alpha n^2}(n\cdot A^a)\partial^2(n\cdot A^a)$$

In this case, the longitudinal part of the vacuum polarization receives contributions proportional to $n^\mu$ in order for the appropriate WTST identity to be satisfied.

The functional integral technique introduced by Slavnov (4) proves to be the most convenient device for determining the WTST identity when one works with the gauge-fixing condition of [1]. The generating functional is given by

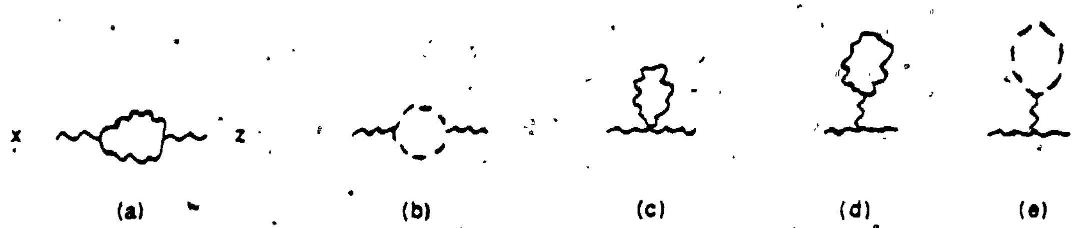
$$[3]\quad Z[\eta_\mu] = \int dA_\mu \Delta(A_\mu) \exp i \int dz \left[-\tfrac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \tfrac{1}{2}(\partial\cdot A - \lambda A^2)^2 + \eta\cdot A\right]$$

Invariance of $Z$ under the gauge transformation

$$[4]\quad A_\mu(x) \to A_\mu(x) - \partial_\mu\psi(x) = A_\mu(x) - \partial_\mu \int dy\, D(x-y)\xi(y) \qquad (\Box\psi = \xi)$$

implies that

$$[5]\quad 0 = \int dA_\mu\, \Delta(A_\mu)\left\{[\partial\cdot A(x) - \lambda A^2(x)] - \int dy\,(\partial\cdot A(y) - \lambda A^2(y))(2\lambda A\cdot\partial D(y-x)) - \int dy\, \eta(y)\cdot\partial D(y-x)\right\} \exp i \int dz\,[\mathcal{L}_{eff} - \eta\cdot A]$$



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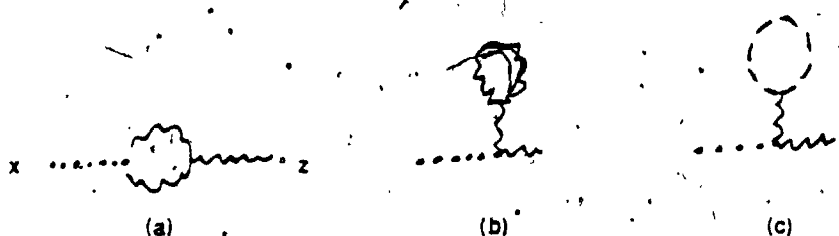

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FIGS. 1–4 Diagrams representing terms to order $\lambda^2$ that contribute to the two-point WTST identity (Wavy lines – photons, dashed lines – ghosts, dotted lines – factors of $D$.)

where

$$\mathcal{L}_{eff} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2}(\partial\cdot A + \lambda A^2)^2$$

(Remember that $dA_\mu \Delta(A_\mu)$ is an invariant measure (4).)

From [5] we arrive at the general form of the WTST identity

$$[6]\quad 0 = -i\frac{\partial}{\partial x^\mu}\frac{\delta Z}{\delta\eta_\mu(x)} - \int dy\,\eta_\mu(y)\frac{\partial}{\partial y_\mu}D(x-y)Z - \lambda\frac{\delta^2 Z}{\delta\eta_\mu(x)\delta\eta^\mu(x)} - 2\lambda\int\frac{\partial}{\partial x_\nu}D(x-y)$$

$$\times\left[-\frac{\delta}{\delta\eta^\nu(y)}\frac{\partial}{\partial y^\mu}\left(\frac{\delta Z}{\delta\eta_\mu(y)}\right) + i\lambda\frac{\delta^3 Z}{\delta\eta_\mu(y)\delta\eta^\mu(y)\delta\eta_\nu(y)}\right]$$

If we take $\delta/\delta\eta_\kappa(z)$ of this equation and set $\eta_\alpha = 0$, then

$$[7]\quad 0 = -i\frac{\partial}{\partial x^\mu}\left[\frac{\delta^2 Z}{\delta\eta_\mu(x)\delta\eta_\kappa(z)}\right]_{\eta=0} - \frac{\partial}{\partial z_\kappa}D(x-z)[Z]_{\eta=0} - \lambda\left[\frac{\delta^3 Z}{\delta\eta_\mu(x)\delta\eta^\mu(x)\delta\eta_\kappa(z)}\right]_{\eta=0}$$

$$-2\lambda\frac{\partial}{\partial x_\nu}\int dy\,D(x-y)\left[-\frac{\delta^2}{\delta\eta_\kappa(z)\delta\eta^\nu(y)}\frac{\partial}{\partial y^\mu}\frac{\delta Z}{\delta\eta_\mu(y)} - i\lambda\frac{\delta^4 Z}{\delta\eta_\kappa(z)\delta\eta_\mu(y)\delta\eta^\mu(y)\delta\eta_\nu(y)}\right]_{\eta=0}$$

From [7] it is seen that the longitudinal part of the vacuum polarization receives a contribution proportional to $\lambda$.

To demonstrate this to order $\lambda^2$ in perturbation theory, we make the standard expansion for $Z$. (The factor $\Delta(A_\mu)$ has been expressed in terms of Faddeev–Popov ghost fields $c$ and $\bar{c}$ (1).)

$$[8]\quad Z[\eta,\bar{\xi},\xi] = \int dA_\mu\,d\bar{c}\,dc\,\exp i\int dz\left[\frac{1}{2}A^\mu\Box A_\mu + \bar{c}\Box c - \lambda(\partial\cdot A)(A^2)\right.$$

$$\left. - \frac{1}{2}\lambda^2(A^2)^2 + 2\lambda\bar{c}A\cdot\partial c + \eta\cdot A + \bar{\xi}c + \bar{c}\xi\right]$$

$$= \exp i\int dx\left[\lambda\left(\frac{\partial}{\partial x^\mu}\frac{\delta}{\delta\eta_\mu(x)}\right)\left(\frac{\delta}{\delta\eta^\nu(x)}\frac{\delta}{\delta\eta_\nu(x)}\right) - \lambda^2\left(\frac{\delta^2}{\delta\eta^\lambda(x)\delta\eta_\lambda(x)}\right)^2\right.$$

$$\left. + 2i\lambda\frac{\delta}{\delta\xi(x)}\frac{\delta}{\delta\eta^\lambda(x)}\frac{\partial}{\partial x_\lambda}\frac{\delta}{\delta\bar{\xi}(x)}\right]\exp -i\int dz\,dw\left[\bar{\xi}(z)D(z-w)\xi(w) + \frac{1}{2}\eta^\mu(z)D(z-w)\eta_\mu(w)\right]$$

Thus, we see that the connected contributions to [7] coming from the first, third, fourth, and fifth terms are given by the diagrams of Figs. 1–4, respectively.

If we use dimensional regularization, then a considerable simplification occurs as integrals of the form $\int d^n k/k^2$ are regulated to zero (5). The only contributions, thus, come from the integrals associated with Figs. 1$a$, 1$b$, 2$a$, and 3$a$. Explicit calculation shows that vacuum polarization, calculated from the integrals associated with Figs. 1$a$ and 1$b$, is given by

$$\Pi_{\mu\nu}(p) = 2\lambda^2(n-2)\int\frac{d^n k}{(2\pi)^n}\int_0^1 dx\,\frac{p_\mu p_\nu}{[k^2 + x(1-x)p^2]^2}$$

The contributions of Figs. 2$a$ and 3$a$, when combined with this result, show that the WTST identity of [7] is indeed satisfied.

## Acknowledgements

We would like to thank the Natural Sciences and Engineering Research Council for financial support, and R. and D. MacKenzie and V. Elias for useful discussions.

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## APPENDIX 3. THE FOUR POINT FUNCTION IN N = 4 SUPERSYMMETRY

In contrast to the ultraviolet divergences which occur in internal loop momentum integrals in connected Green's functions for Abelian and non-Abelian gauge field theories, in supersymmetric field theories, such functions are devoid of this kind of infinity. In fact, it has been shown by Mandelstam ( 1), Howe, Stelle and Townsend ( 2) and Brink, Schwarz and Scherk ( 3) that the N=4 Super Yang-Mills theory is free of all ultraviolet divergences. Thus, all infinities, if they exist, must be of an infrared nature.

For the purpose of studying the infrared behaviour of connected Green's functions in such a theory, a calculation of the one loop chiral, four point connected Green's function, corresponding to the amplitude for $\phi\ \phi \rightarrow \bar{\phi}\ \bar{\phi}$ scattering, in massless N=4 Super Yang-Mills theory, using the formalism presented by Grisaru, Rocek and Siegel ( 4), is presented in this appendix. This work was done in collaboration with Gerry McKeon and Subhash Rajpoot from King's College, the University of London.

For an arbitrary gauge group, where all of the fields are in the adjoint representation, the superspace action in this theory has the following form:

$$S = \mathrm{tr}\Big\{\int d^4x d^4\theta \; e^{-gV}\,\bar{\phi}_\alpha e^{gV}\,\phi_\alpha + (1/64g^2)\int d^4x d^4\theta\, W^A W_A$$
$$+ (ig/6)\int d^4x d^2\theta\; \varepsilon_{\alpha\beta\gamma}\phi_\alpha\{\phi_\beta,\phi_\gamma\}_-$$
$$+ (ig/6)\int d^4x d^2\bar{\theta}\; \varepsilon_{\alpha\beta\gamma}\bar{\phi}_\alpha\{\bar{\phi}_\beta,\bar{\phi}_\gamma\}_-\Big\}. \tag{1}$$

In equation (1) the indices, $\alpha$, on the chiral (and anti-chiral) scalar superfields are internal symmetry indices and the indices, A, on the superfield strength are spinor indices;

$$V = V^a T^a \text{ and } \phi_\alpha = \phi^a_\alpha T^a \tag{2a}$$

where the matrices $T^a$ are a specific representation of the generators of the gauge group satisfying the standard group requirement that,

$$\{T^a, T^b\}_- = i f^{abc} T^c \tag{2b}$$

and the vector superfield strength spinor, $W^A$, is defined in terms of the supercovariant derivatives, $D_A$ and $\bar{D}_{\dot{A}}$, in the following manner:

$$W_A \equiv \bar{D}^2(e^{-gV} D_A e^{gV}) \tag{2c}$$

The chiral and anti-chiral massless scalar superfields satisfy the respective dynamical equations:

$$D_A(p,\theta)\bar{\phi}_\alpha(p,\theta) = 0 \tag{3a}$$

and

$$\bar{D}_{\dot{A}}(p,\theta)\phi_\alpha(p,\theta) = 0 \tag{3b}$$

where, in momentum space,

$$D_A(p,\theta) \equiv (\partial/\partial\theta^A) - \bar{\theta}^{\dot{A}} p_{A\dot{A}} \tag{4a}$$

and

$$\bar{D}_{\dot{A}}(p,\theta) \equiv -(\partial/\partial\bar{\theta}^{\dot{A}}) + \theta^A p_{A\dot{A}} \tag{4b}$$

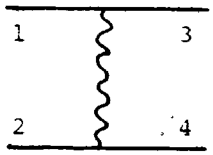
where p is the momentum flowing <u>out of the point</u> on which the derivatives act. Furthermore, in order for the chiral (and anti-chiral) scalar superfields to satisfy the equations of motion, the condition of masslessness implies the additional restrictions that

$$D^2(p,\theta)\phi_\alpha(p,\theta) = \bar{D}^2(p,\theta)\bar{\phi}_\alpha(p,\theta) = 0 \tag{5a}$$

and

$$p^{A\dot{A}}D_A(p,\theta)\phi_\alpha(p,\theta) = p^{A\dot{A}}\bar{D}_{\dot{A}}(p,\theta)\bar{\phi}_\alpha(p,\theta) = 0 \tag{5b}$$

Employing the supersymmetric Feynman gauge, the necessary 1PI Feynman rules are summarized by Caswell and Zanon ( 5) and by Grisaru, Rocek and Siegel ( 6). As shown by Grisaru, Rocek and Siegel ( 4) all one loop diagrams which contain a simple self energy insertion on either a scalar or a vector superfield line are zero. This result follows directly from the supersymmetry algebra. The entire set of diagrams which contribute to the massless, chiral scalar superfield four point connected Green's function in N=4 Super Yang-Mills theory is shown in Figure 1. The direction of all momenta is to the right. The diagrams in Figure 1 which contain vertex corrections only can be most easily calculated using the results for the one loop, 1PI Green's functions calculated by Grisaru, Roček and Siegel ( 4). The explicit diagrammatical form of these vertices, which in Figure 1 are represented by a solid circle, are indicated in Figure 2.

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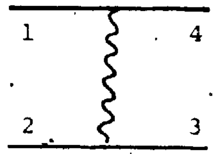


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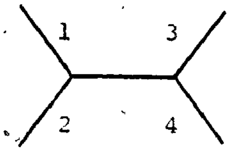


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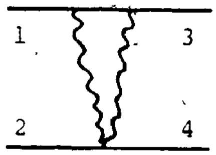


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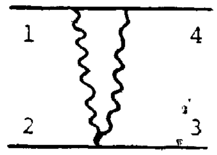


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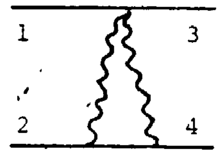


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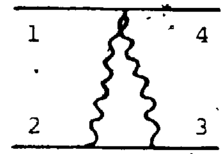


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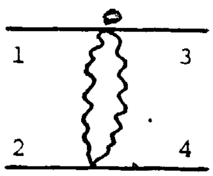


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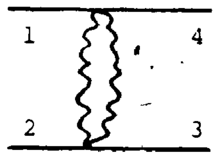


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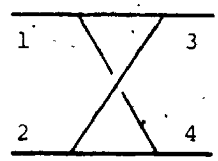


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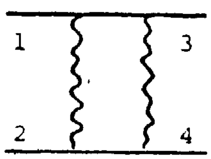


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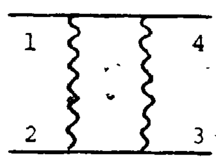


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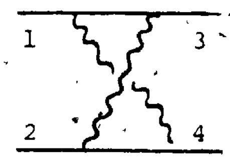


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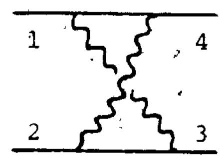


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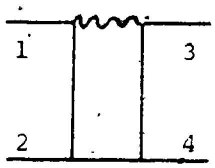


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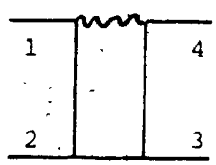


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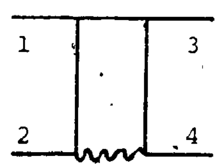


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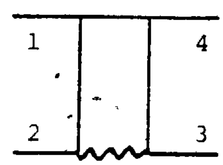


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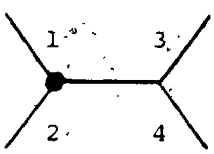


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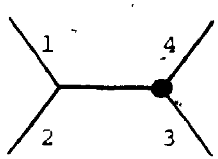


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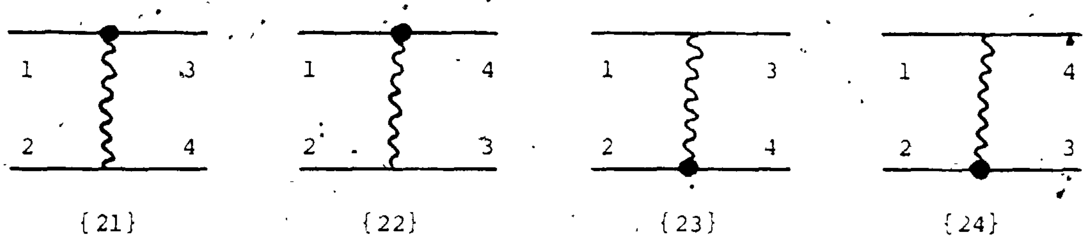


Figure 1. A diagrammatic representation of the contributions to the S matrix element corresponding to $\phi\ \phi \to \bar{\phi}\ \bar{\phi}$ scattering. The number 1 on an external line refers to the fact that the scalar superfield $\phi_{\alpha}^{a}(-p_1,\theta)$ is attached to that line. Similarly this notation applies to the numbers 2, 3 and 4.

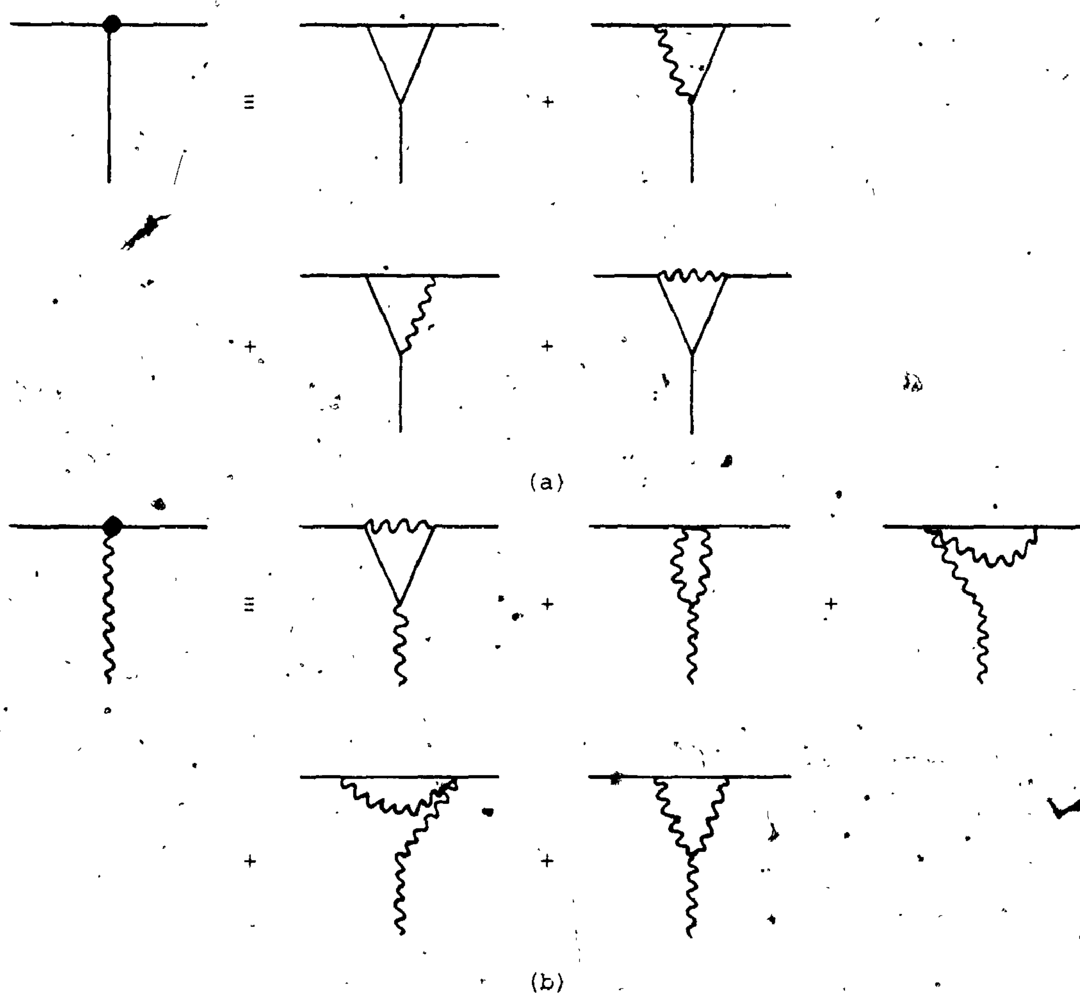


Figure 2. A diagrammatic representation of the one loop corrections to the three point superfield vertices. Figure 2(a) involves the three point chiral (or anti-chiral) scalar superfield vertex and Figure 2(b) involves the anti-chiral scalar-chiral scalar-vector superfield vertex.

According to the commutation and anti-commutation rules for the D operators that are given in the appendix of Caswell and Zanon ( 5), in calculating the S matrix element that corresponds to the four point connected Green's function under consideration, the following groupings of D operators, in terms of their simplest forms, occur:

$$\delta_{12}\delta_{12} = 0 \tag{6a}$$

$$\delta_{12}(D_1^2\delta_{12}\overleftarrow{\bar{D}_2^2}) = \delta_{12}(D_1^2\bar{D}_1^2\delta_{12}) = 16\delta_{12} \tag{6b}$$

$$\begin{aligned}
&\int d^4\theta_1 d^4\theta_2 d^4\theta_3 d^4\theta_4\ \phi_\alpha^a(-p_1,\theta_1)\phi_\beta^b(-p_2,\theta_2)\bar{\phi}_\gamma^c(p_3,\theta_3)\bar{\phi}_\delta^d(p_4,\theta_4)\\
&\qquad\times\ \delta_{12}\delta_{34}(D_1^2\delta_{13}\overleftarrow{\bar{D}_3^2})(D_2^2\delta_{24}\overleftarrow{\bar{D}_4^2})\\
&\quad= (16)^2\int d^4\theta\ \phi_\alpha^a(-p_1,\theta)\phi_\beta^b(-p_2,\theta)\bar{\phi}_\gamma^c(p_3,\theta)\bar{\phi}_\delta^d(p_4,\theta)
\end{aligned} \tag{6c}$$

$$\begin{aligned}
&\int d^4\theta_1 d^4\theta_2 d^4\theta_3 d^4\theta_4\ \Big\{\phi_\alpha^a(-p_1,\theta_1)\phi_\beta^b(-p_2,\theta_2)\bar{\phi}_\gamma^c(p_3,\theta_3)\bar{\phi}_\delta^d(p_4,\theta_4)\\
&\qquad\times\ \delta_{13}\delta_{24}(D_1^2(k-p_1)\delta_{14}\overleftarrow{\bar{D}_4^2}(k-p_1))(D_2^2(k-p_3)\delta_{23}\overleftarrow{\bar{D}_3^2}(k-p_3))\Big\}\\
&\quad= 16\int d^4\theta\ \Big\{(\phi_\beta^b(-p_2,\theta)\bar{\phi}_\delta^d(p_4,\theta))\{D^2\bar{D}^2-8(k-p_1)^{A\dot{A}}D_A\bar{D}_{\dot{A}}\\
&\qquad\qquad -16(k-p_1)^2\}(\phi_\alpha^a(-p_1,\theta)\bar{\phi}_\gamma^c(p_3,\theta))\Big\}
\end{aligned} \tag{6d}$$

In equation (6), and all equations with similar content, the $\theta$ dependence of a given supercovariant derivative is indicated by a simple subscript. The momentum dependence of such a super-covariant derivative is either,

i) implicitly given by the same subscript as the $\theta$ variable,

ii) indicated explicitly, as is done in the left hand side of equation (6d), or

iii) assumed, when operating on a product of superfields (chiral and/or anti-chiral), to be the sum of the individual momentum variables of each superfield.

Thus, in the right hand side of equation (6d), the following content is implied:

$$D^2\bar{D}^2 = D^2(-p_1+p_3,\theta)\bar{D}^2(-p_1+p_3,\theta)$$

The following results are obtained for the individual contributions to the S matrix element for $\phi\,\phi \to \bar{\phi}\,\bar{\phi}$ scattering that are depicted in Figure 1.

$$\{1\} = g^2\Gamma f^{acs}f^{bds}\delta_{\alpha\gamma}\delta_{\beta\delta}\{1/(p_3-p_1)^2\} \tag{7a}$$

$$\{2\} = g^2\Gamma f^{ads}f^{bcs}\delta_{\alpha\delta}\delta_{\beta\gamma}\{1/(p_4-p_1)^2\} \tag{7b}$$

$$\{3\} = g^2\Gamma f^{abs}f^{cds}(\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma})\{1/(p_1+p_2)^2\} \tag{7c}$$

$$\{4\} = \{g^4/(2\pi)^n\}\Gamma\{(\Lambda^{adbc}+\Lambda^{abdc})/2\}\delta_{\alpha\gamma}\delta_{\beta\delta}I_3(p_1,p_3) \tag{7d}$$

$$\{5\} = \{g^4/(2\pi)^n\}\Gamma\{(\Lambda^{abcd}+\Lambda^{acbd})/2\}\delta_{\alpha\delta}\delta_{\beta\gamma}I_3(p_1,p_4) \tag{7e}$$

$$\{6\} = \{g^4/(2\pi)^n\}\Gamma\{(\Lambda^{abdc}+\Lambda^{adbc})/2\}\delta_{\alpha\gamma}\delta_{\beta\delta}I_3(p_2,p_4) \tag{7f}$$

$$\{7\} = \{g^4/(2\pi)^n\}\Gamma\{(\Lambda^{abcd}+\Lambda^{acbd})/2\}\delta_{\alpha\delta}\delta_{\beta\gamma}I_3(p_2,p_4) \tag{7g}$$

$$\{8\} = 0 \tag{7h}$$

$$\{9\} = 0 \tag{7i}$$

$$\begin{aligned}\{10\} = {}& (1/4)\{g^4/(2\pi)^n\}\Lambda^{acbd}(\delta_{\alpha\delta}\delta_{\beta\gamma}+\delta_{\alpha\gamma}\delta_{\beta\delta}) \\ & \times \int \bar{d}^4p_1\bar{d}^4p_2\bar{d}^4p_3\bar{d}^4p_4 d^4\theta\,\{\phi^b_\beta(-p_2,\theta)\bar{\phi}^c_\gamma(p_3,\theta)\} \\ & \times \{(1/16)D^2\bar{D}^2I_4(-p_4,p_1,-p_3)-(1/2)I_4^{AB}(-p_4,p_1,-p_3)D_A\bar{D}_B \\ & -I_3(-p_3,-p_1)\}\{\phi^a_\alpha(-p_1,\theta)\bar{\phi}^d_\delta(p_4,\theta)\}\end{aligned} \tag{7j}$$

$$\{11\} = -\{g^4/(2\pi)^n\}\Gamma\Lambda^{abdc}\delta_{\alpha\gamma}\delta_{\beta\delta}(p_1+p_2)^2I_4(p_2,-p_4,-p_3) \tag{7k}$$

$$\{12\} = -\{g^4/(2\pi)^n\}\Gamma\Lambda^{abcd}\delta_{\alpha\delta}\delta_{\beta\gamma}(p_1+p_2)^2I_4(p_2,-p_3,-p_4) \tag{7l}$$

$$\begin{aligned}\{13\} = {}& \{g^4/(2\pi)^n\}\Lambda^{acbd}\delta_{\alpha\gamma}\delta_{\beta\delta}\int \bar{d}^4p_1\bar{d}^4p_2\bar{d}^4p_3\bar{d}^4p_4 d^4\theta \\ & \times \{\phi^b_\beta(-p_2,\theta)\bar{\phi}^c_\gamma(p_3,\theta)\}\{(1/16)D^2\bar{D}^2I_4(p_1,-p_4,p_2) \\ & -(1/2)I_4^{AB}(p_1,-p_4,p_2)D_A\bar{D}_B-I_3(p_2,p_4)\} \\ & \times \{\phi^a_\alpha(-p_1,\theta)\bar{\phi}^d_\delta(p_4,\theta)\}\end{aligned} \tag{7m}$$

$$\{14\} = \{g^4/(2\pi)^n\}\Lambda^{adbc}\delta_{\alpha\delta}\delta_{\beta\gamma}\int d^4p_1 d^4p_2 d^4p_3 d^4p_4 d^4\theta$$
$$\times \{\phi^b_\beta(-p_2,\theta)\bar{\phi}^d_\delta(p_4,\theta)\}\{(1/16)D^2\bar{D}^2 I_4(p_1,-p_3,p_2)$$
$$-(1/2)I_4^{A\dot{B}}(p_1,-p_3,p_2)D_A\bar{D}_{\dot{B}} - I_3(p_2,p_3)\}$$
$$\times \{\phi^a_\alpha(-p_1,\theta)\bar{\phi}^c_\gamma(p_3,\theta)\} \qquad (7n)$$

$$\{15\} = \{g^4/(2\pi)^n\}\Lambda^{abdc}(\delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\gamma}\delta_{\beta\delta})\int d^4p_1 d^4p_2 d^4p_3 d^4p_4 d^4\theta$$
$$\times \{\phi^b_\beta(-p_2,\theta)\bar{\phi}^d_\delta(p_4,\theta)\}\{(1/16)D^2\bar{D}^2 I_4(p_1,-p_3,-p_4)$$
$$-(1/2)I_4^{A\dot{B}}(p_1,-p_3,-p_4)D_A\bar{D}_{\dot{B}} - I_3(p_3,-p_4)\}$$
$$\times \{\phi^a_\alpha(-p_1,\theta)\bar{\phi}^c_\gamma(p_3,\theta)\} \qquad (7o)$$

$$\{16\} = \{g^4/(2\pi)^n\}\Lambda^{abcd}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma})\int d^4p_1 d^4p_2 d^4p_3 d^4p_4 d^4\theta$$
$$\times \{\phi^b_\beta(-p_2,\theta)\bar{\phi}^c_\gamma(p_3,\theta)\}\{(1/16)D^2\bar{D}^2 I_4(p_1,-p_4,-p_3)$$
$$-(1/2)I_4^{A\dot{B}}(p_1,-p_4,-p_3)D_A\bar{D}_{\dot{B}} - I_3(p_4,-p_3)\}$$
$$\times \{\phi^a_\alpha(-p_1,\theta)\bar{\phi}^d_\delta(p_4,\theta)\} \qquad (7p)$$

$$\{17\} = \{g^4/(2\pi)^n\}\Lambda^{badc}(\delta_{\beta\delta}\delta_{\alpha\gamma} - \delta_{\beta\gamma}\delta_{\alpha\delta})\int d^4p_1 d^4p_2 d^4p_3 d^4p_4 d^4\theta$$
$$\times \{\phi^a_\alpha(-p_1,\theta)\bar{\phi}^d_\delta(p_4,\theta)\}\{(1/16)D^2\bar{D}^2 I_4(p_2,-p_3,-p_4)$$
$$-(1/2)I_4^{A\dot{B}}(p_2,-p_3,-p_4)D_A\bar{D}_{\dot{B}} - I_3(p_3,-p_4)\}$$
$$\times \{\phi^b_\beta(-p_2,\theta)\bar{\phi}^c_\gamma(p_3,\theta)\} \qquad (7q)$$

$$\{18\} = \{g^4/(2\pi)^n\}\Lambda^{bacd}(\delta_{\beta\gamma}\delta_{\alpha\delta} - \delta_{\alpha\gamma}\delta_{\beta\delta})\int d^4p_1 d^4p_2 d^4p_3 d^4p_4 d^4\theta$$
$$\times \{\phi^a_\alpha(-p_1,\theta)\bar{\phi}^c_\gamma(p_3,\theta)\}\{(1/16)D^2\bar{D}^2 I_4(p_2,-p_4,-p_3)$$
$$-(1/2)I_4^{A\dot{B}}(p_2,-p_4,-p_3)D_A\bar{D}_{\dot{B}} - I_3(p_4,-p_3)\}$$
$$\times \{\phi^b_\beta(-p_2,\theta)\bar{\phi}^d_\delta(p_4,\theta)\} \qquad (7r)$$

$$\{19+20\} = \{g^4/(2\pi)^n\}K_1\Gamma f^{abs}f^{cds}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma})I_3(-p_1,p_2) \qquad (7s+7t)$$

$$\{21\} = -\{g^4/(2\pi)^n\}(K_1/16)f^{acs}f^{bds}\delta_{\alpha\gamma}\delta_{\beta\delta}I_3(p_1,p_3)$$
$$\times \int d^4p_1 d^4p_2 d^4p_3 d^4p_4 d^4\theta\ \{1/(p_3-p_1)^2\}\{\phi^a_\alpha(-p_1,\theta)\bar{\phi}^c_\gamma(p_3,\theta)\}$$
$$\times \{D^A\bar{D}^2 D_A + (p_1+p_3)^{A\dot{B}}\{D_A,\bar{D}_{\dot{B}}\}_-\}\{\phi^b_\beta(-p_2,\theta)\bar{\phi}^d_\delta(p_4,\theta)\} \qquad (7u)$$

$$\{22\} = -\{g^4/(2\pi)^n\}(K_1/16) f^{ads} f^{bcs} \delta_{\alpha\delta}\delta_{\beta\gamma} I_3(p_1,p_4)$$
$$\times \int d^4p_1 d^4p_2 d^4p_3 d^4p_4 d^4\theta \; \{1/(p_4-p_1)^2\}\{\phi^a_\alpha(-p_1,\theta)\overline{\phi}^d_\delta(p_4,\theta)\}$$
$$\times \{D^A\overline{D}^2 D_A + (p_1+p_4)^{A\dot{B}}\{D_A,\overline{D}_{\dot{B}}\}_-\}\{\phi^b_\beta(-p_2,\theta)\overline{\phi}^c_\gamma(p_3,\theta)\} \tag{7v}$$
$$\{23\} = -\{g^4/(2\pi)^n\}(K_1/16) f^{acs} f^{bds} \delta_{\alpha\gamma}\delta_{\beta\delta} I_3(p_2,p_4)$$
$$\times \int d^4p_1 d^4p_2 d^4p_3 d^4p_4 d^4\theta \; \{1/(p_3-p_1)^2\}\{\phi^b_\beta(-p_2,\theta)\overline{\phi}^d_\delta(p_4,\theta)\}$$
$$\times \{D^A\overline{D}^2 D_A + (p_2+p_4)^{A\dot{B}}\{D_A,\overline{D}_{\dot{B}}\}_-\}\{\phi^a_\alpha(-p_1,\theta)\overline{\phi}^c_\gamma(p_3,\theta)\} \tag{7w}$$
$$\{24\} = -\{g^4/(2\pi)^n\}(K_1/16) f^{ads} f^{bcs} \delta_{\alpha\delta}\delta_{\beta\gamma} I_3(p_2,p_3)$$
$$\times \int d^4p_1 d^4p_2 d^4p_3 d^4p_4 d^4\theta \; \{1/(p_4-p_1)^2\}\{\phi^b_\beta(-p_2,\theta)\overline{\phi}^c_\gamma(p_3,\theta)\}$$
$$\times \{D^A\overline{D}^2 D_A + (p_2+p_3)^{A\dot{B}}\{D_A,\overline{D}_{\dot{B}}\}_-\}\{\phi^a_\alpha(-p_1,\theta)\overline{\phi}^d_\delta(p_4,\theta)\} \tag{7x}$$

In equation (7) the following definitions have been made.

$$\Lambda^{abcd} \equiv f^{wax} f^{xby} f^{ycz} f^{zdw} \tag{8a}$$

$$(1/2)K_1 f^{abc} \equiv f^{xay} f^{ybz} f^{zcx} \tag{8b}$$

and

$$\Gamma \equiv \int d^4p_1 d^4p_2 d^4p_3 d^4p_4 d^4\theta \; \phi^a_\alpha(-p_1,\theta)\phi^b_\beta(-p_2,\theta)\overline{\phi}^c_\gamma(p_3,\theta)\overline{\phi}^d_\delta(p_4,\theta) \tag{8c}$$

The integrals that arise in calculating these diagrams and which appear in equation (7) have the following generic form:

$$I_3(p_1,{}_2) \equiv \int d^nk \; \{1/(k^2(k+p_1)^2(k+p_2)^2)\} \tag{9a}$$

$$I_4(p_1,p_2,p_3) \equiv \int d^nk \; \{1/(k^2(k+p_1)^2(k+p_1+p_2)^2(k+p_1+p_2+p_3)^2)\} \tag{9b}$$

and

$$I_4^\mu(p_1,p_2,p_3) \equiv \int d^nk \; \{k^\mu/(k^2(k+p_1)^2(k+p_1+p_2)^2(k+p_1+p_2+p_3)^2)\} \tag{9c}$$

Making use of the technique of dimensional regularization to parametrize the infinities, the following perturbative results are obtained for the integrals defined in equation (9):

$$I_3(p_1,p_2) = -\{i\pi^2/2p_{12}\varepsilon^2\}\{1+\varepsilon(-\gamma-\ln(2\pi p_{12}))+\varepsilon^2((1/2)\ln^2(2\pi p_{12})$$
$$+\gamma\ln(2\pi p_{12})+((\gamma^2-\zeta)/2))+O(\varepsilon^3)\} \qquad (10a)$$

$$I_4(p_1,p_2,p_3) = \{i\pi^2/p_{12}p_{23}\varepsilon^2\}\{1+\varepsilon(-\gamma-(1/2)\ln(4\pi^2 p_{12}p_{23}))$$
$$+\varepsilon^2\{((\gamma^2-\zeta)/2)+(\gamma/2)\ln(4\pi^2 p_{12}p_{23})$$
$$-(1/2)\ln(p_{12}/p_{23})\ln(-p_{12}/p_{23})$$
$$+(1/4)\ln^2(-2\pi p_{12})+(1/4)\ln^2(-2\pi p_{23})$$
$$-(1/2)\mathrm{Sp}(-p_{13}/p_{12})-(1/2)\mathrm{Sp}(-p_{13}/p_{23})\}$$
$$+O(\varepsilon^3)\} \qquad (10b)$$

and

$$I_4^\mu(p_1,p_2,p_3) = A(p_1,p_2,p_3)(2p_1+p_2+p_3)^\mu$$
$$+ B(p_1,p_2,p_3)(p_1+p_2)^\mu \qquad (11a)$$

where

$$A(p_1,p_2,p_3) = -\{i\pi^2/4p_{12}p_{23}\varepsilon^2\}\{1-\varepsilon(\gamma+\ln(-2\pi p_{12}))$$
$$+\varepsilon^2\{((\gamma^2-\zeta)/2)+\gamma\ln(-2\pi p_{12})+(1/2)\ln^2(-2\pi p_{12})$$
$$+(p_{12}/p_{13})\{\ln(p_{23}/p_{12})\ln(-p_{23}/p_{12})$$
$$+\mathrm{Sp}(-p_{13}/p_{12})+\mathrm{Sp}(-p_{13}/p_{23})\}\}$$
$$+O(\varepsilon^3)\} \qquad (11b)$$

and

$$B(p_1,p_2,p_3) = -\{i\pi^2/4p_{12}p_{23}\varepsilon^2\}\{1-\varepsilon(\gamma+\ln(-2\pi p_{23}))$$
$$+\varepsilon^2\{((\gamma^2-\zeta)/2)+\gamma\ln(-2\pi p_{23})+(1/2)\ln^2(-2\pi p_{23})$$
$$+(p_{23}/p_{13})\{\ln(p_{12}/p_{23})\ln(-p_{12}/p_{23})$$
$$+\mathrm{Sp}(-p_{13}/p_{23})+\mathrm{Sp}(-p_{13}/p_{12})\}\}$$
$$+O(\varepsilon^3)\} \qquad (11c)$$

In equations (10) and (11),

$$\varepsilon \equiv 2-n/2 \qquad (12a)$$

$$p_{ij} \equiv p_i \cdot p_j \qquad (12b)$$

and

$$k^{\mu} \equiv k^{A\dot{A}}\sigma^{\mu}_{A\dot{A}}. \tag{12c}$$

In order to eliminate the explicit dependence of certain integrals, given in equation (7), on the supercovariant derivatives, it is extremely useful to consider the general quantity, $\Omega(-p_1,-p_2,p_3,p_4,\theta)$, defined in the following manner:

$$\Omega(-p_1,-p_2,p_3,p_4,\theta) \equiv (Ap_1+Bp_2+Cp_3+Dp_4)^{A\dot{B}}\{D_A(-p_1-p_2+p_3+p_4,\theta) \times \phi^a_\alpha(-p_1,\theta)\phi^b_\beta(-p_2,\theta)\bar{\phi}^c_\gamma(p_3,\theta)\{\bar{D}_{\dot{B}}(p_4,\theta)\bar{\phi}^d_\delta(p_4,\theta)\}\} \tag{13}$$

Making use of the anti-commutation relation:

$$\{D_A(k,\theta),\bar{D}_{\dot{B}}(k,\theta)\}_+ = 2k_{A\dot{B}} \tag{14}$$

and the chirality conditions given in equation (3), the function defined in equation (13) has the following form:

$$\begin{aligned}\Omega(-p_1,-p_2,p_3,p_4,\theta) = {} & (Ap_1+Bp_2+Cp_3+Dp_4)^{A\dot{B}} \\ & \times\Big\{(\{D_A(-p_1,\theta)\phi^a_\alpha(-p_1,\theta)\}\phi^b_\beta(-p_2,\theta) \\ & \quad \times \bar{\phi}^c_\gamma(p_3,\theta)\{\bar{D}_{\dot{B}}(p_4,\theta)\bar{\phi}^d_\delta(p_4,\theta)\}) \\ & +(\phi^a_\alpha(-p_1,\theta)\{D_A(-p_2,\theta)\phi^b_\beta(-p_2,\theta)\} \\ & \quad \times \bar{\phi}^c_\gamma(p_3,\theta)\{\bar{D}_{\dot{B}}(p_4,\theta)\bar{\phi}^d_\delta(p_4,\theta)\}) \\ & +\phi^a_\alpha(-p_1,\theta)\phi^b_\beta(-p_2,\theta)\bar{\phi}^c_\gamma(p_3,\theta) \\ & \quad \times \{2p_{4,A\dot{B}}\bar{\phi}^d_\delta(p_4,\theta)\}\Big\}\end{aligned} \tag{15}$$

Employing the restrictions, that are valid for a massless theory, given in equation (5b), equation (15) has the form:

$$\begin{aligned}\Omega(-p_1,-p_2,p_3,p_4,\theta) = {} & \Big\{(Bp_2+Cp_3)^{A\dot{B}}(\{D_A(-p_1,\theta)\phi^a_\alpha(-p_1,\theta)\}\phi^b_\beta(-p_2,\theta) \\ & \times \bar{\phi}^c_\gamma(p_3,\theta)\{\bar{D}_{\dot{B}}(p_4,\theta)\bar{\phi}^d_\delta(p_4,\theta)\}) \\ & +(Ap_1+Cp_3)^{A\dot{B}}(\phi^a_\alpha(-p_1,\theta)\{D_A(-p_2,\theta)\phi^b_\beta(-p_2,\theta)\} \\ & \times \bar{\phi}^c_\gamma(p_3,\theta)\{\bar{D}_{\dot{B}}(p_4,\theta)\bar{\phi}^d_\delta(p_4,\theta)\})\end{aligned}$$

$$+4p_4\cdot(Ap_1+Bp_2+Cp_3)\phi^a_\alpha(-p_1,\theta)\phi^b_\beta(-p_2,\theta) \times \overline{\phi}^c_\gamma(p_3,\theta)\overline{\phi}^d_\delta(p_4,\theta)\Big\} \tag{16}$$

since

$$p_i^2 = 0 \ ; \ i = 1,2,3,4 \tag{17a}$$

and

$$p^{A\dot{A}}q_{A\dot{A}} = 2p\cdot q \tag{17b}$$

By virtue of the fact that, for a physical process, $-p_1-p_2+p_3+p_4=0$, the supercovariant derivative, $D_A(-p_1-p_2+p_3+p_4,\theta)$, occurring in equation (13) has the form:

$$D_A(-p_1-p_2+p_3+p_4,\theta) = (\partial/\partial\theta^A) - \overline{\theta}^{\dot{A}}(-p_1-p_2+p_3+p_4)_{A\dot{A}} \tag{18a}$$

$$= (\partial/\partial\theta^A) \tag{18b}$$

Thus, the integral of $\Omega(-p_1,-p_2,p_3,p_4,\theta)$ over all $\theta$ space vanishes identically since it is just the integral of the total derivative of a function that vanishes at the end points of the integration region.

Taking the specific values for the constants,

$A = C = 0$ and $B = 1$,

and integrating over all $\theta$ leads to the following result:

$$0 = \int d^4\theta \Big\{p_2^{A\dot{B}}\{D_A(-p_1,\theta)\phi^a_\alpha(-p_1,\theta)\}\phi^b_\beta(-p_2,\theta)\overline{\phi}^c_\gamma(p_3,\theta)\{\overline{D}_{\dot{B}}(p_4,\theta)\overline{\phi}^d_\delta(p_4,\theta)\} + 4p_{24}\phi^a_\alpha(-p_1,\theta)\phi^b_\beta(-p_2,\theta)\overline{\phi}^c_\gamma(p_3,\theta)\overline{\phi}^d_\delta(p_4,\theta)\Big\} \tag{19}$$

By having the supercovariant derivative, $\overline{D}_{\dot{B}}$, in equation (13), act on the anti-chiral scalar superfield, $\overline{\phi}^c_\gamma(p_3,\theta)$, and by choosing different values for the constants, A, B, C and D, in the same equation, results similar to that given in equation (19) can be obtained. This allows all of the explicit supercovariant derivative dependence in equation (7) to be eliminated.

The contributions to the S matrix given in equation (7) which contain this explicit dependence have, following this analysis, the form:

$$\{10\} = (1/4)\{g^4/(2\pi)^n\}\Gamma\Lambda^{acbd}(\delta_{\alpha\delta}\delta_{\beta\gamma}+\delta_{\alpha\gamma}\delta_{\beta\delta})\{-2p_{14}B(-p_4,p_1,-p_3) - I_3(-p_3,-p_1)\} \quad (20a)$$

$$\{13\} = \{g^4/(2\pi)^n\}\Gamma\Lambda^{acbd}\delta_{\alpha\gamma}\delta_{\beta\delta}\{-2p_{14}(2A(p_1,-p_4,p_2)+B(p_1,-p_4,p_2)) - I_3(p_2,p_4)\} \quad (20b)$$

$$\{14\} = \{g^4/(2\pi)^n\}\Gamma\Lambda^{adbc}\delta_{\alpha\delta}\delta_{\beta\gamma}\{-2p_{13}(2A(p_1,-p_3,p_2)+B(p_1,-p_3,p_2)) - I_3(p_2,p_3) \quad (20c)$$

$$\{15\} = \{g^4/(2\pi)^n\}\Gamma\Lambda^{abdc}(\delta_{\alpha\delta}\delta_{\beta\gamma}-\delta_{\alpha\gamma}\delta_{\beta\delta})\{-2p_{13}(B(p_1,-p_3,-p_4) + A(p_1,-p_3,-p_4)) - I_3(p_3,-p_4)\} \quad (20d)$$

$$\{16\} = \{g^4/(2\pi)^n\}\Gamma\Lambda^{abcd}(\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma})\{-2p_{14}(B(p_1,-p_4,-p_3) + A(p_1,-p_4,-p_3)) - I_3(p_4,-p_3)\} \quad (20e)$$

$$\{17\} = \{g^4/(2\pi)^n\}\Gamma\Lambda^{badc}(\delta_{\beta\delta}\delta_{\alpha\gamma}-\delta_{\beta\gamma}\delta_{\alpha\delta})\{-2p_{23}(B(p_2,-p_3,-p_4) + A(p_2,-p_3,-p_4)) - I_3(p_3,-p_4)\} \quad (20f)$$

$$\{18\} = \{g^4/(2\pi)^n\}\Gamma\Lambda^{bacd}(\delta_{\beta\gamma}\delta_{\alpha\delta}-\delta_{\beta\delta}\delta_{\alpha\gamma})\{-2p_{24}(B(p_2,-p_4,-p_3) + A(p_2,-p_4,-p_3)) - I_3(p_4,-p_3)\} \quad (20g)$$

$$\{21\} = (1/4)\{g^4/(2\pi)^n\}K_1\Gamma f^{acs}f^{bds}\delta_{\alpha\gamma}\delta_{\beta\delta}I_3(p_1,p_3) \quad (20h)$$

$$\{22\} = (1/4)\{g^4/(2\pi)^n\}K_1\Gamma f^{ads}f^{bcs}\delta_{\alpha\delta}\delta_{\beta\gamma}I_3(p_1,p_4) \quad (20i)$$

$$\{23\} = (1/4)\{g^4/(2\pi)^n\}K_1\Gamma f^{bds}f^{acs}\delta_{\alpha\gamma}\delta_{\beta\delta}I_3(p_2,p_4) \quad (20j)$$

$$\{24\} = (1/4)\{g^4/(2\pi)^n\}K_1\Gamma f^{bcs}f^{ads}\delta_{\alpha\delta}\delta_{\beta\gamma}I_3(p_2,p_3) \quad (20k)$$

Summing the contributions from all of the diagrams depicted in Figure 1 yields the S matrix element corresponding to $\phi\ \phi \rightarrow \bar{\phi}\ \bar{\phi}$ scattering, a result which contains single and double poles in $\varepsilon$, the parameter specifying the deviation from four dimensions of the spacetime manifold. As there are no ultraviolet divergences in one loop diagrams in this theory, a fact that is easily seen by power counting, all of the singular structure contained in this S matrix element must be of an infrared nature. In the context of dimensional regularization, Lee and Milgram ( 7) have outlined a method of separating the total divergence in a given loop integral into its infrared and ultraviolet divergent parts. This analysis is applicable to the integrals defined in equation (9).

One possible method of interpreting this infrared problem is to employ the analysis suggested by Sterman and Weinberg ( 8), in the study of jets in Q.C.D., and use the energy scattered outside of a cone of fixed solid angle as an infrared cut-off. As discussed by Juer and Storey ( 9), massless supersymmetric field theories have severe infrared problems for physical, on mass-shell amplitudes. Again, this kind of behaviour could be suspected on the basis of simple power counting considerations, since the ultraviolet finiteness of such a theory implies that there are many powers of momentum in the denominators of the loop integrals.

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## APPENDIX 4. A DETAILED CALCULATION OF THE ONE LOOP FERMION FIELD TWO POINT CONNECTED GREEN'S FUNCTION

Referring to equation (3.17), in which a general n-point connected Green's function is defined, the two point fermion field connected Green's function has the form:

$$\langle 0|T(\bar{\phi}_m(x)\phi_n(y))|0\rangle_c$$

$$= \frac{\left\{(-\delta/i\delta K_{1m}(x))(\delta/i\delta\bar{K}_{1n}(y))\right\}\exp\{i\int d^4x\, L_g\}\exp\{i\tau\}}{\exp\{i\int d^4x\, L_g\}\exp\{i\tau\}} \tag{1}$$

where all of the sources are set equal to zero.

All of the terms in the quantity $\tau$, the value of which is given in equation (3.18), contain two source functions only, and there are no terms which couple fermionic and bosonic sources to each other. As a result of these properties, when the limit as all of these sources tend to zero is taken, only the terms in the expansion of the factor $\exp\{i\int d^4x\, L_g\}$, in equation (1), that contain an even number of functional differentiations with respect to both the fermionic and bosonic source functions will contribute to this particular connected Green's function.

To second order in the coupling constant the relevant terms in this expansion are

$$\exp\{i\int d^4w\, L_g\}$$

$$= 1 \qquad (2a)$$

$$- (ig^2/2)\int d^4w\, \overline{K}_1(w)\Lambda(w)\Lambda(w)\psi(w) \qquad (2b)$$

$$- (ig^2/2)\int d^4w\, \overline{\psi}(w)\Lambda(w)\Lambda(w)K_1(w) \qquad (2c)$$

$$+ ig^2\int d^4w\, A^{\alpha}(w)\{\Gamma_{\alpha\beta}(\partial/\partial w^{\lambda})A^{\beta}(w)\} \qquad (2d)$$

$$+ ig^2\int d^4w\, \overline{\psi}(w)\{\Sigma(\partial/\partial w^{\lambda})\psi(w)\} \qquad (2e)$$

$$- (g^2/2)\int d^4w d^4z\, \overline{\psi}(w)\not{A}(w)\psi(w)\overline{\psi}(z)\not{A}(z)\psi(z) \qquad (2f)$$

$$- ig^2\int d^4w d^4z\, \overline{\psi}(w)\not{A}(w)\psi(w)\overline{K}_1(z)\Lambda(z)\psi(z) \qquad (2g)$$

$$+ ig^2\int d^4w d^4z\, \overline{\psi}(w)\not{A}(w)\psi(w)\overline{\psi}(z)\Lambda(z)K_1(z) \qquad (2h)$$

$$- g^2\int d^4w d^4z\, \overline{\psi}(w)\Lambda(w)K_1(w)\overline{K}_1(z)\Lambda(z)\psi(z) \qquad (2i)$$

In equation (2), for the sake of brevity, the functions $\Lambda(w)$, $\psi_i(w)$, $\overline{\psi}_j(w)$ and $A_{\mu}(w)$ are understood to stand for the associated functional derivatives. For example,

$$\psi_i(w) = \delta/i\delta\overline{K}_{2i}(w) \qquad (3)$$

The contribution from the term in equation (2g) to the numerator of the two point connected fermion field Green's function is the following:

$$\xi_{mn}(x,y) \equiv \overline{\phi}_m(x)\phi_n(y)(-ig^2)\int d^4w d^4z\, \overline{\psi}(w)\not{A}(w)\psi(w)\overline{K}_1(z)\Lambda(z)\psi(z) \times \exp\{i\tau\} \qquad (4a)$$

In equation (4a) the function $\overline{\phi}_m(x)$ stands for the functional derivative:

$$\overline{\phi}_m(x) = -\delta/i\delta K_{1m}(x) \qquad (4b)$$

and likewise for $\phi_n(y)$.

An explicit evaluation of the quantity $\xi_{mn}(x,y)$, defined in equation (4a), is presented below. The terms which vanish when all of the source functions are set equal to zero will be ignored at all stages of this calculation.

$$\xi_{mn}(x,y) = -ig^2\int d^4w d^4z\ \overline{\phi}_m(x)\phi_n(y)\overline{\psi}_i(w)\gamma^\alpha_{ij}A_\alpha(w)\psi_j(w) \times \overline{K}_{lk}(z)\Lambda(z)\psi_k(z)\exp\{i\tau\} \tag{5a}$$

$$= -ig^2\int d^4w d^4z\ \gamma^\alpha_{ij}A_\alpha(w)\Lambda(z)\overline{\phi}_m(x)\overline{\psi}_i(w)\psi_j(w) \times \{\phi_n(y)\overline{K}_{lk}(z)\}\psi_k(z)\exp\{i\tau\} \tag{5b}$$

By definition,

$$\phi_n(y)\overline{K}_{lk}(z) = (1/i)\delta_{nk}\delta^4(z-y) \tag{6}$$

and therefore,

$$\xi_{mn}(x,y) = -g^2\int d^4w\ \gamma^\alpha_{ij}A_\alpha(w)\Lambda(y)\overline{\phi}_m(x)\overline{\psi}_i(w)\psi_j(w)\psi_n(y)\exp\{i\tau\} \tag{7a}$$

$$= -g^2\int d^4w\ \gamma^\alpha_{ij}A_\alpha(w)\Lambda(y)\overline{\phi}_m(x)\overline{\psi}_i(w) \times \int d^4p d^4q\ \{S_{jr}(w,p)K_{1r}(p)S_{nl}(y,q)K_{2l}(q) + S_{jr}(w,p)K_{2r}(p)S_{nl}(y,q)K_{1l}(q)\}\exp\{i\tau\} \tag{7b}$$

$$= g^2\int d^4w\ \gamma^\alpha_{ij}A_\alpha(w)\Lambda(y)\{\delta/\delta K_{1m}(x)\} \times \int d^4p d^4q\ \{-S_{jr}(w,p)K_{1r}(p)S_{nl}(y,q)\delta_{il}\delta^4(q-w) + S_{jr}(w,p)\delta_{ri}\delta^4(p-w)S_{nl}(y,q)K_{1l}(q)\}\exp\{i\tau\} \tag{7c}$$

The second term on the right hand side of equation (7c) vanishes as a result of the fact that:

$$\gamma^\alpha_{ij}S_{ji}(w,w) = 0 \tag{8}$$

This is shown most easily by considering the Fourier transform representation of the quantity:

$$\gamma^\alpha_{ij}S_{ji}(w,w) = \int d^4p\ e^{-ip\cdot(w-w)}\{\gamma^\alpha_{ij}(\not p+m)_{ji}\}/\{p^2-m^2\} \tag{9a}$$

$$= \int d^4p\ \{4p^\alpha/(p^2-m^2)\} \tag{9b}$$

since $\qquad (\gamma^\alpha\gamma^\beta)_{ii} = 4g^{\alpha\beta}$ $\qquad$ (9c)

and $\qquad (\gamma^\alpha)_{ii} = 0$ $\qquad$ (9d)

The desired result follows immediately from equation (9b) since an odd integrand is being integrated over a symmetric interval.

Returning to equation (7c) and performing the functional differentiation that is explicitly indicated leads to the following result:

$$\xi_{mn}(x,y) = -g^2\int d^4w\ \gamma^{\alpha}_{ij}A_{\alpha}(w)\Lambda(y)S_{jm}(w,x)S_{ni}(y,w)\exp\{i\tau\} \qquad (10a)$$

$$= -g^2\int d^4w\ S_{ni}(y,w)\gamma^{\alpha}_{ij}S_{jm}(w,x)\{i(\partial/\partial w^{\alpha})F(w,y)\} \times \exp\{i\tau\} \qquad (10b)$$

Eliminating the factor of $\exp\{i\tau\}$ in equation (10b) leads to the following final result:

$$\xi_{mn}(x,y) = -ig^2\int d^4w \left\{\{(\partial/\partial w^{\mu})F(w,y)\}S_{ni}(y,w)\gamma^{\mu}_{ij}S_{jm}(w,x)\right\} \qquad (11)$$

This is just the contribution to the connected Green's function under consideration that is given in equation (4.18f).

In order to obtain the actual value of the divergent contribution to the connected Green's function from this quantity it is worthwhile to replace the propagators in equation (11) with their Fourier transform representations:

$$\xi_{mn}(x,y) = -ig^2\int d^4w\int d\!\!\!{}^{-}{}^4p\ e^{-ip\cdot(w-y)}(-ip_{\mu})\{1/(p^2)^2\} \times \int d\!\!\!{}^{-}{}^4q\ e^{-iq\cdot(y-w)}\{(\not{q}+m)_{ni}/(q^2-m^2)\}\gamma^{\mu}_{ij} \times \int d\!\!\!{}^{-}{}^4r\ e^{-ir\cdot(w-x)}\{(\not{r}+m)_{jm}/(r^2-m^2)\} \qquad (12a)$$

$$= -g^2\int d\!\!\!{}^{-}{}^4p\,d^4q\,d\!\!\!{}^{-}{}^4r \left\{p_{\mu}\{1/(p^2)^2\}\{(\not{q}+m)_{ni}/(q^2-m^2)\} \times \gamma^{\mu}_{ij}\{(\not{r}+m)_{jm}/(r^2-m^2)\}\int d\!\!\!{}^{-}{}^4w\ e^{-iw\cdot(p-q+r)} \times e^{ip\cdot y}e^{-iq\cdot y}e^{ir\cdot x}\right\} \qquad (12b)$$

$$= -g^2\int d\!\!\!{}^{-}{}^4p\,d^4q\,d\!\!\!{}^{-}{}^4r \left\{p_{\mu}\{1/(p^2)^2\}\{(\not{q}+m)_{ni}/(q^2-m^2)\} \times \gamma^{\mu}_{ij}\{(\not{r}+m)_{jm}/(r^2-m^2)\}\delta^4(p-q+r)e^{ip\cdot y}e^{-iq\cdot y}e^{ir\cdot x}\right\}$$

(12c)

$$\xi_{mn}(x,y) = -g^2 \int d^4r\, e^{-ir\cdot(y-x)}\{(\not{r}+m)_{jm}/(r^2-m^2)\}$$
$$\times \int d^4p \left\{\{(\not{p}+\not{r}+m)\not{p}\}_{nj}/(p^2)^2\{(p+r)^2-m^2\}\right\} \qquad (12d)$$

Making use of equations (4.14h) and (4.14i), the ultraviolet divergent integral in equation (12d) has the singular structure:

$$\int d^4p \left\{\{(\not{p}+\not{r}+m)\not{p}\}_{nj}/(p^2)^2\{(p+r)^2-m^2\}\right\}$$
$$= \{i/8\pi^2\}\delta_{nj}\ln(\Lambda/\mu) + \text{finite} \qquad (13)$$

Inserting this result into equation (12d) yields the following, final value for the divergent part of $\xi_{mn}(x,y)$:

$$\xi^{div.}_{mn}(x,y) = -g^2 \int d^4q\, e^{-iq\cdot(y-x)}\{(\not{q}+m)_{nm}/(q^2-m^2)\}\{i/8\pi^2\}\ln(\Lambda/\mu) \qquad (14a)$$

$$= -I_{nm}(y,x) \qquad (14b)$$

where the quantity $I_{nm}(y,x)$ is defined in equation (4.20).

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March 18, 1986

TO WHOM IT MAY CONCERN:

This is to permit the paper "Pre-Regularization for Supersymmetry" by V. Elias, G. McKeon, S.B. Phillips and R.B. Mann, published in Physics Letters, Volume 133B, number 1,2 (1983) to be reprinted as an appendix in the thesis of Dr. S.B. Phillips.

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TO WHOM IT MAY CONCERN:

This is to permit the paper "Longitudinal contributions to the vacuum polarization in the 't Hooft-Veltman gauge" by R.B. Mann, G. McKeon and S. Phillips, published in Can. J. Phys. 62, 1129 (1934) to be reprinted as an appendix in the thesis of Dr. S.B. Phillips.

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